October 10 Math 2306 sec. 51 Fall 2022

Section 10: Variation of Parameters

$$y'' + P(x)y' + Q(x)y = g(x)$$

If $\{y_1, y_2\}$ is a fundamental solution set for the associated homogeneous equation, then the general solution is

$$y = y_c + y_p$$

where

$$y_c = c_1 y_1(x) + c_2 y_2(x), \quad \text{and} \quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Letting W denote the Wronskian of y_1 and y_2, the functions u_1 and u_2 are given by the formulas

$$u_1 = \int \frac{-y_2 g}{W}\,dx, \quad \text{and} \quad u_2 = \int \frac{y_1 g}{W}\,dx.$$

Find the general solution

$$y'' - 2y' + y = \frac{e^x}{1+x^2} + x$$

$y = y_c + y_p$

Find y_c: y_c solves $y'' - 2y' + y = 0$

Characteristic eqn $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \implies m = 1$ double root

$\underline{y_1 = e^x, \quad y_2 = xe^x}$

$$\boxed{y_c = c_1 e^x + c_2 x e^x}$$

$$y'' - 2y' + y = \frac{e^x}{1+x^2} + x$$

We can look for $y_p = y_{p_1} + y_{p_2}$

where y_{p_1} solves

$$y'' - 2y' + y = \frac{e^x}{1+x^2}$$

and y_{p_2} solves $y'' - 2y' + y = x$

Find y_{p_1} using Variation of parameters

$$y_1 = e^x, \quad y_2 = xe^x, \quad g(x) = \frac{e^x}{1+x^2}$$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^x(xe^x + e^x) - e^x(xe^x)$$

$$= xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$$

$$u_1 = \int \frac{-g\,y_2}{W}\,dx = -\int \frac{\left(\frac{e^x}{1+x^2}\right) x e^x}{e^{2x}}\,dx$$

$$= -\int \frac{\frac{x e^{2x}}{1+x^2}}{e^{2x}}\,dx = -\int \frac{x}{1+x^2}\,dx$$

Let $v = 1+x^2$, $dv = 2x\,dx \Rightarrow \frac{1}{2}\,dv = x\,dx$

$$u_1 = \frac{-1}{2}\int \frac{dv}{v} = \frac{-1}{2}\ln|v| = \frac{-1}{2}\ln(1+x^2)$$

$$u_2 = \int \frac{g\,y_1}{W}\,dx = \int \frac{\frac{e^x}{1+x^2}\,e^x}{e^{2x}}\,dx.$$

$$= \int \frac{dx}{1+x^2} = \tan^{-1} x$$

$$y_1 = e^x, \quad y_2 = x e^x, \quad u_1 = \frac{-1}{2}\ln(1+x^2), \quad u_2 = \tan^{-1} x$$

$$\boxed{\begin{aligned} y_{p_1} &= u_1 y_1 + u_2 y_2 \\ &= \frac{-1}{2} e^x \ln(1+x^2) + x e^x \tan^{-1} x \end{aligned}}$$

Find y_{p_2} $\qquad y'' - 2y' + y = x$

Let's use undetermined coefficients

$g_2(x) = x$ $\qquad$ 1^{st} degree polynomial

$y_{p_2} = Ax + B$

$$y_{p_2}' = A$$
$$y_{p_2}'' = 0$$

$$y_{p_2}'' - 2y_{p_2}' + y_{p_2} = x$$

$$0 - 2(A) + (Ax + B) = x$$

$$\underline{\underline{Ax}} + \underline{\underline{(-2A + B)}} = \underline{\underline{x}} + \underline{\underline{0}}$$

Matching : $A = 1$, $-2A + B = 0 \Rightarrow B = 2A = 2$

$$\boxed{y_{p_2} = x + 2.}$$

$$\boxed{y_c = c_1 e^x + c_2 x e^x}$$

$$\boxed{\begin{aligned} y_{p_1} &= u_1 y_1 + u_2 y_2 \\ &= -\frac{1}{2} e^x \ln(1+x^2) + x e^x \tan^{-1} x \end{aligned}}$$

$$y_p = y_{p_1} + y_{p_2} \quad \text{and} \quad y = y_c + y_p$$

The general solution

$$\boxed{y = c_1 e^x + c_2 x e^x - \frac{1}{2} e^x \ln(1+x^2) + x e^x \tan^{-1} x + x + 2}$$

Section 11: Linear Mechanical Equations

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion**.

Harmonic Motion gif

Building an Equation: Hooke's Law

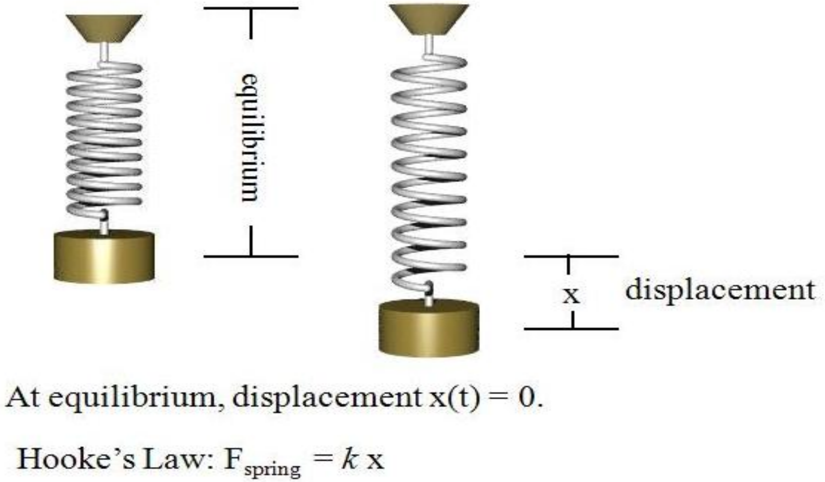

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ (mass times acceleration)

$$a = \frac{d^2x}{dt^2} \quad \Longrightarrow \quad F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ (proportional to displacement)

Equating forces

$$mx'' = -kx \quad \Longrightarrow \quad mx'' + kx = 0$$

this is a 2nd order, linear, constant coefficient homogeneous equation.

Standard form: $x'' + \frac{k}{m}x = 0$ Let $\omega^2 = \frac{k}{m}$

$$\boxed{x'' + \omega^2 x = 0}$$

Displacment in Equilibrium

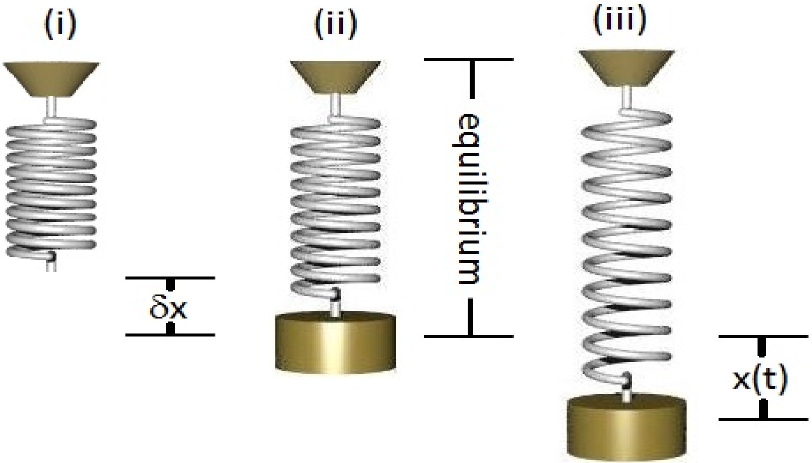

Figure: Spring only, versus spring-mass equilibrium, and spring-mass (nonzero) displacement

Obtaining the Spring Constant (US Customary Units)

If an object with weight W pounds stretches a spring δx feet in equilibrium, then by Hooke's law we compute the spring constant via the equation

$$W = k\delta x. \quad \Rightarrow \quad k = \frac{W}{\delta x}$$

The units for k in this system of measure are lb/ft.

$$k = \frac{W}{\delta x} \qquad \frac{\text{lb}}{\text{ft}}$$

Obtaining the Spring Constant (US Customary Units)

Note also that Weight = mass $\times$ acceleration due to gravity. Hence if we know the weight of an object, we can obtain the mass via

$$W = mg.$$

We typically take the approximation $g = 32$ ft/sec^2. The units for mass are lb sec^2/ft which are called slugs.

$m = \dfrac{W}{g}$ the units are slugs

Obtaining the Spring Constant (SI Units)

In SI units,

- Weight (force) would be in Newtons (N),
- Length would be in meters (m),
- Spring constant would be in N/m
- Mass would be in kilograms (kg)

It is customary to describe an object by its mass in kilograms. When we encounter such a description, we deduce the weight in Newtons

$$W = mg \quad \text{taking the approximation} \quad g = 9.8\,\text{m}/\text{sec}^2.$$

The *Circular Frequency* ω

Applying Hooke's law with the weight as force, we have

$$\frac{mg}{m\delta x} = \frac{k\delta x.}{m\delta x} \Rightarrow \frac{g}{\delta x} = \frac{k}{m}$$

We observe that the value ω can be deduced from δx by

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

Provided that values for δx and g are used in appropriate units, ω is in units of per second.

Simple Harmonic Motion

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) \tag{2}$$

called the **equation of motion.**

Caution: The phrase **equation of motion** is used differently by different authors.

Some use this phrase to refer the IVP (1). Others use it to refer to the **solution** to the IVP such as (2).

We'll use the convention up $x>0$, down $x<0$.

Simple Harmonic Motion

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[1]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[1] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

Amplitude and Phase Shift (alternative definition)

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \cos(\omega t - \hat{\phi})$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and this **phase shift** $\hat{\phi}$ must be defined by

$$\cos \hat{\phi} = \frac{x_0}{A}, \quad \text{with} \quad \sin \hat{\phi} = \frac{x_1}{\omega A}.$$