# October 10 Math 3260 sec. 51 Fall 2025

## Chapter 4 Vector Spaces & Subspaces

In this chapter, we will

- learn about additional properties of vectors in $R^n$,
- learn about special subsets of $R^n$, including some related to matrices,
- state the **Fundamental Theorem of Linear Algebra**,
- and pin down precisely what a **vector space** is.

# 4.1 Linear Independence

**Definition: Linear Independence**

The collection of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $R^m$ is said to be **linearly independent** if the homogeneous equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}_m \tag{1}$$

has only the trivial solution, $x_1 = x_2 = \cdots = x_n = 0$.

If the collection of vectors is not linearly independent, then we say that it is **linearly dependent**.

For a linearly dependent set of vectors, an equation of the form (1) having at least one nonzero weight is called a **linear dependence relation**.

# Some Observations on Linear (In)dependence

- Every nonempty set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in $R^m$ is either linearly independent or linearly dependent.

- Linear independence/dependence is a property of a set (or collection) of vectors.

  *"The column vectors of A are linearly dependent."* (makes sense)

  *"The matrix A is linearly dependent."* (doesn't make sense)

- We saw last time that a set containing one vector, $\{\vec{v}\}$, in $R^m$ is **linearly**

$$\begin{cases} \textbf{independent} \text{ if} & \vec{v} \neq \vec{0}_m \\ \textbf{dependent} \text{ if} & \vec{v} = \vec{0}_m \end{cases}$$

  So $\{\langle 1, 0, 2, 1\rangle\}$ is linearly independent, while $\{\langle 0, 0, 0\rangle\}$ is linearly dependent.

The set $\{\vec{e}_1, \vec{e}_2\}$ in $R^2$ is linearly independent.

Consider $x_1\vec{e}_1 + x_2\vec{e}_2 = \vec{0}_2$

$$x_1\langle 1,0\rangle + x_2\langle 0,1\rangle = \langle 0,0\rangle$$

$$\langle x_1,0\rangle + \langle 0,x_2\rangle = \langle 0,0\rangle$$

$$\langle x_1, x_2\rangle = \langle 0,0\rangle$$

Comparing entries, $x_1=0$ and $x_2=0$.

$x_1\vec{e}_1 + x_2\vec{e}_2 = \vec{0}_2$ has only the trivial solution. So $\{\vec{e}_1, \vec{e}_2\}$ is linearly independent.

Show that the set $\{\langle 1,0,1\rangle, \langle -3,0,-3\rangle\}$ is linearly dependent.

Consider $x_1\langle 1,0,1\rangle + x_2\langle -3,0,-3\rangle = \langle 0,0,0\rangle$.

$$\langle x_1-3x_2,\ 0,\ x_1-3x_2\rangle = \langle 0,0,0\rangle$$

This holds for any $x_1, x_2$ such that

$x_1-3x_2=0$, i.e., $x_1=3x_2$

There are non trivial solutions. An example of a linear dependence relation is

$$3\langle 1,0,1\rangle + 1\langle -3,0,-3\rangle = \langle 0,0,0\rangle.$$

So $\{\langle 1,0,1\rangle, \langle -3,0,-3\rangle\}$ is linearly dependent.

# A Set of Two Vectors

A set of two vectors, $\{\vec{v}_1, \vec{v}_2\}$, in $R^n$ is linearly dependent if and only if one of the vectors is a scalar multiple of the other.

Let's show that if one is a scalar multiple of the other, $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent.

Suppose $\vec{v}_1 = c\vec{v}_2$ for some scalar $c$.

Rearranging gives

$$\vec{v}_1 - c\vec{v}_2 = \vec{0}_n$$

This is a linear dependence relation because the coefficient of $\vec{v}_1$ is $1 \neq 0$. So $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent.

Let's show that if $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent, then one is a scalar multiple of the other.

Suppose $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent. There there exist scalars $x_1$ and $x_2$, not both zero, such that

$$x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{0}_n$$

Assuming $x_1 \neq 0$, then

$$x_1\vec{v}_1 = -x_2\vec{v}_2$$

$$\vec{v}_1 = \frac{-x_2}{x_1}\vec{v}_2$$

Since $\frac{-x_2}{x_1}$ is some scalar, we see that $\vec{v}_1$ is a scalar multiple of $\vec{v}_2$.

# Example

Identify each set as being linearly dependent or linearly independent.

1. $\{\langle 1, 2, 1\rangle\}$ Lin. Independent.

2. $\{\langle 4, 2, -1, 0\rangle, \langle -8, -4, 2, 0\rangle\}$ Lin. dependent

3. $\{\langle 1, 1\rangle, \langle 0, 0\rangle\}$ Lin. dependent, $\langle 0,0\rangle = 0\langle 1,1\rangle$

4. $\{\langle 1, 3, 0, 4\rangle, \langle 2, 0, 6, 8\rangle\}$ Lin. Independent.

## Three or More Vectors

With a set of three or more vectors, we can always turn an equation like

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}_m$$

into a matrix-vector equation

$$A\vec{x} = \vec{0}_m$$

by setting

$$\mathrm{Col}_i(A) = \vec{v}_i, \quad i = 1, \ldots, n.$$

**Example:** Determine whether the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent or linearly independent where

$$\vec{v}_1 = \langle -2, 4, -5\rangle, \quad \vec{v}_2 = \langle -5, 8, -6\rangle, \quad \vec{v}_3 = \langle 3, 0, -12\rangle.$$

$\vec{v}_1 = \langle -2, 4, -5 \rangle, \quad \vec{v}_2 = \langle -5, 8, -6 \rangle, \quad \vec{v}_3 = \langle 3, 0, -12 \rangle$

Set up $A\vec{x} = \vec{0}_3$ where the columns of $A$ are the $\vec{v}$'s. Let $A$ have columns

$\text{Col}_i(A) = \vec{v}_i$.

$$[A \mid \vec{0}_3] = \left[\begin{array}{ccc|c} -2 & -5 & 3 & 0 \\ 4 & 8 & 0 & 0 \\ -5 & -6 & -12 & 0 \end{array}\right] \xrightarrow{\text{rref}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$A\vec{x} = \vec{0}_3$ has nontrivial solutions $\Rightarrow$ $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

$\vec{v}_1 = \langle -2, 4, -5 \rangle, \quad \vec{v}_2 = \langle -5, 8, -6 \rangle, \quad \vec{v}_3 = \langle 3, 0, -12 \rangle$

Not only can we conclude that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, this rref can be used to form a linear dependence relation. Turns out, the relationship between the first three columns of the rref is identical to the first three columns of the original matrix. That is,

$$\vec{v}_3 = 6\vec{v}_1 + (-3)\vec{v}_2$$

This means that

$$6\vec{v}_1 - 3\vec{v}_2 - \vec{v}_3 = \vec{0}_3 \text{ is}$$

a lin. dependence relation.