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Chapter 4 Vector Spaces & Subspaces

In this chapter, we will

- learn about additional properties of vectors in R^n,
- learn about special subsets of R^n, including some related to matrices,
- state the **Fundamental Theorem of Linear Algebra**,
- and pin down precisely what a **vector space** is.

4.1 Linear Independence

Definition: Linear Independence

The collection of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in R^m is said to be **linearly independent** if the homogeneous equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}_m \qquad (1)$$

has only the trivial solution, $x_1 = x_2 = \cdots = x_n = 0$.

If the collection of vectors is not linearly independent, then we say that it is **linearly dependent**.

For a linearly dependent set of vectors, an equation of the form (1) having at least one nonzero weight is called a **linear dependence relation**.

Some Observations on Linear (In)dependence

- Every nonempty set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in R^m is either linearly independent or linearly dependent.

- Linear independence/dependence is a property of a set (or collection) of vectors.

  "The column vectors of A are linearly dependent." (makes sense)

  "The matrix A is linearly dependent." (doesn't make sense)

- We saw last time that a set containing one vector, $\{\vec{v}\}$, in R^m is **linearly**

$$\begin{cases} \textbf{independent} \text{ if} & \vec{v} \neq \vec{0}_m \\ \textbf{dependent} \text{ if} & \vec{v} = \vec{0}_m \end{cases}$$

  So $\{\langle 1, 0, 2, 1\rangle\}$ is linearly independent, while $\{\langle 0, 0, 0\rangle\}$ is linearly dependent.

The set $\{\vec{e}_1, \vec{e}_2\}$ in R^2 is linearly independent.

Show that the set $\{\langle 1, 0, 1\rangle, \langle -3, 0, -3\rangle\}$ is linearly dependent.

A Set of Two Vectors

A set of two vectors, $\{\vec{v}_1, \vec{v}_2\}$, in R^n is linearly dependent if and only if one of the vectors is a scalar multiple of the other.

Example

Identify each set as being linearly dependent or linearly independent.

1. $\{\langle 1, 2, 1\rangle\}$

2. $\{\langle 4, 2, -1, 0\rangle, \langle -8, -4, 2, 0\rangle\}$

3. $\{\langle 1, 1\rangle, \langle 0, 0\rangle\}$

4. $\{\langle 1, 3, 0, 4\rangle, \langle 2, 0, 6, 8\rangle\}$

Three or More Vectors

With a set of three or more vectors, we can always turn an equation like

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}_m$$

into a matrix-vector equation

$$A\vec{x} = \vec{0}_m$$

by setting

$$\mathrm{Col}_i(A) = \vec{v}_i, \quad i = 1, \dots, n.$$

Example: Determine whether the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent or linearly independent where

$$\vec{v}_1 = \langle -2, 4, -5\rangle, \quad \vec{v}_2 = \langle -5, 8, -6\rangle, \quad \vec{v}_3 = \langle 3, 0, -12\rangle.$$

$$\vec{v}_1 = \langle -2, 4, -5 \rangle, \quad \vec{v}_2 = \langle -5, 8, -6 \rangle, \quad \vec{v}_3 = \langle 3, 0, -12 \rangle$$

Matrix Columns

Theorem: Let A be an $m \times n$ matrix. The column vectors of A are linearly independent in R^m if and only if the homogeneous equation $A\vec{x} = \vec{0}_m$ has only the trivial solution.

Corollary: Square Matrices & Invertibility

If A is an $n \times n$ matrix, then A is invertible if and only if the column vectors of A are linearly independent.

Remark: Since the invertibility of A implies invertibility of A^T, we can also say that A is invertible if and only if the **row** vectors of A are linearly independent.

Some Linearly Dependent Sets

Theorem: Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a collection of k of vectors in R^n. If

a. one of the vectors, say $\vec{v}_i = \vec{0}_n$, or if

b. $k > n$,

then the collection is linearly dependent.

Remark: Note what this says. It says

- Any set that includes a zero vector is automatically linearly dependent.
- If a set contains more vectors than there are entries in each vector, it's automatically linearly dependent.

Example

Explain why each set below is linearly dependent.

1. $\{\langle 0, 0\rangle, \langle 1, 2\rangle\}$

2. $\{\langle 1, -3\rangle, \langle 5, 7\rangle, \langle 4, -1\rangle\}$

Subsets of R^n

A **subset** of R^n is just some collection of vectors in R^n. We can come up with tons of examples of subsets:

- $B = \{\vec{e}_1, \vec{e}_2\}$ in R^2 is a subset containing the two vectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$.
- The set $W = \{\langle a, b, 0\rangle \mid a, b \in R\}$ is the subset of R^3 of all vectors whose last entry is zero.
- The set $T = \{\langle k, n\rangle \mid k, n \in Z\}$ is the subset of R^2 of all vectors having integer entries.
- The set $P = \mathrm{Span}\{\langle 1, 0, 1, 0\rangle\}$ is the subset of R^4 of all scalar multiples of $\langle 1, 0, 1, 0\rangle$.

4.2 Subspaces of R^n

Not all subsets are created equal. Recall that we have two critical operations in R^n:

- vector addition and
- scalar multiplication.

Subsets of R^n that hold a special sort of importance in Linear Algebra are sets that in some sense preserve these two operations. Such sets are *similar* to R^n in that we can do arithmetic with these operations in the set.

Definition: Subspace of R^n

A subset, S, of R^n is a **subspace** of R^n provided

i. S is nonempty,

ii. for any pair of vectors $\vec{u}$ and $\vec{v}$ in S, $\vec{u} + \vec{v}$ is in S, and

iii. for any vector $\vec{u}$ in S and scalar c in R, $c\vec{u}$ is in S.

A set that satisfies property

ii. is said to be **closed with respect to vector addition**.

iii. is said to be **closed with respect to scalar multiplication**.

The phrase *"with respect to"* can be repaced with the word *"under"*.

Example: $V = \{\langle a, b\rangle \mid a, b \in R \text{ and } ab \geq 0\}$.

Which of the following vectors is in V?

1. $\langle 2, 3\rangle$
2. $\langle 4, -2\rangle$
3. $\langle -5, -2\rangle$
4. $\langle 0, 0\rangle$
5. $\langle -12, 0\rangle$
6. $\langle -6, 8\rangle$

In the standard, Cartesian coordinate system, a vector in V would have to have standard representation in which quadrant(s)? (I, II, III, or IV)

$V = \{\langle a, b\rangle \mid a, b \in R \text{ and } ab \geq 0\}$.

Suppose $\langle a, b\rangle$ is in V. Is the scalar multiple $c\langle a, b\rangle$ in V is

1. $c > 0$?

2. $c < 0$?

3. $c = 0$?

$V = \{\langle a, b\rangle \mid a, b \in R \text{ and } ab \geq 0\}$.

The following vectors are all in V:

$$\begin{array}{lllllllll} \vec{x}_1 & = & \langle 2,3\rangle, & \vec{x}_3 & = & \langle 0,0\rangle, & \vec{x}_5 & = & \langle -5,-2\rangle, \\ \vec{x}_2 & = & \langle 1,1\rangle, & \vec{x}_4 & = & \langle -1,-1\rangle, & \vec{x}_6 & = & \langle -6,0\rangle, \end{array}$$

Which of the following sums are in V?

$\vec{x}_1 + \vec{x}_2$

$\vec{x}_3 + \vec{x}_4$

$\vec{x}_5 + \vec{x}_6$

$\vec{x}_1 + \vec{x}_5$

$\vec{x}_2 + \vec{x}_4$

$\vec{x}_2 + \vec{x}_6$

$V = \{\langle a, b\rangle \mid a, b \in R \text{ and } ab \geq 0\}$.

1. Is V a nonempty subset of R^2?

2. Is V closed under scalar multiplication?

3. Is V closed under vector addition?

4. Is V a subspace of R^2?

Example

Determine whether the following set is a subspace of R^3.

$$W = \{\langle a, b, 0\rangle \mid a, b \in R\}.$$

Observation

Note that for any vector $\vec{x} = \langle a, b, 0 \rangle$ in W,

$$\vec{x} = \langle a, b, 0 \rangle = a\langle 1, 0, 0 \rangle + b\langle 0, 1, 0 \rangle.$$

So

$$\vec{x} \in \mathrm{Span}\left\{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle\right\}.$$

Based on how **span** is defined, a set that is defined as a span is closed under both vector addition and scalar multiplication.

Theorem

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is any nonempty subset of vectors in R^n, then the set $\mathrm{Span}(S)$ is a subspace of R^n.

Remark: Another way to prove that some set is a subspace of R^n is to show that it's a span.

Example

Show that the set G defined below is a subspace of R^4 by finding a spanning set.

$$G = \{\langle a+b, a, -b, 0\rangle \mid a, b \in R\}$$

4.2.1 Fundamental Subspaces of a Matrix

We will define four subspaces, two of R^m and two of R^n, associated with an $m \times n$ matrix. Collectively, we call these the **Fundamental Subspaces of a Matrix**.

Column Space

Let A be an $m \times n$ matrix. The subspace of R^m spanned by the column vectors of A, denoted

$$\mathcal{CS}(A) = \mathrm{Span}\{\mathrm{Col}_1(A), \ldots, \mathrm{Col}_n(A)\},$$

is called the **column space of** A.

Remark: We can say

"$\mathcal{CS}(A)$ is the set of all vectors $\vec{y} \in R^m$ such that $A\vec{x} = \vec{y}$ is consistent."

Row Space

Let A be an $m \times n$ matrix. The subspace of R^n spanned by the row vectors of A, denoted

$$\mathcal{RS}(A) = \text{Span}\{\text{Row}_1(A), \ldots, \text{Row}_m(A)\},$$

is called the **row space of A**.

Remark: There is a geometric interpretation of $\mathcal{RS}(A)$, but it will make more sense after we define another fundamental subspace.

Example

Let $A = \begin{bmatrix} 0 & 3 & 1 \\ 4 & 7 & 5 \\ -2 & -5 & -3 \\ 5 & -4 & 2 \end{bmatrix}$. Identify a spanning set for $\mathcal{RS}(A)$ and a spanning set for $\mathcal{CS}(A)$.

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 4 & 7 & 5 \\ -2 & -5 & -3 \\ 5 & -4 & 2 \end{bmatrix}$$

Example

Characterize the column and row spaces of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

$\mathcal{CS}(A)$ & $\mathcal{RS}(A)$ of $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

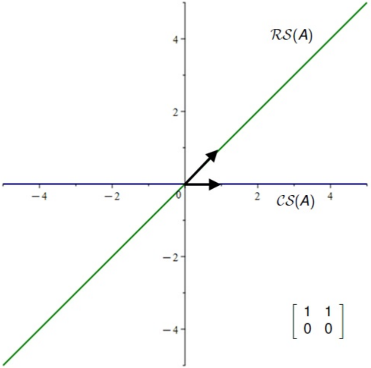

Figure: The row and column spaces of this matrix A are the lines $x_2 = x_1$ and $x_2 = 0$, respectively.

A Third Fundamental Subspace

Definition: Null Space

The **null space** of A, denoted $\mathcal{N}(A)$, is the set of all solutions of the homogeneous equation $A\vec{x} = \vec{0}_m$. That is,

$$\mathcal{N}(A) = \{\vec{x} \in R^n \mid A\vec{x} = \vec{0}_m\}.$$

Theorem

Let A be an $m \times n$ matrix. Then $\mathcal{N}(A)$ is a subspace of R^n.

Proof:

For an $m \times n$ matrix A, we have to show that (1) $\mathcal{N}(A)$ is not empty, (2) $\mathcal{N}(A)$ is closed under vector addition, and (3) $\mathcal{N}(A)$ is closed under scalar multiplication.

Example

Find a spanning set for $\mathcal{N}(A)$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Example

Find a spanning set for $\mathcal{N}(A^T)$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Interpreting the Fundamental Subspaces

We already have an interpretation of the column space and the null space. For $m \times n$ matrix A

- $\mathcal{CS}(A)$ is all $\vec{y} \in R^m$ such that $A\vec{x} = \vec{y}$ is consistent, and
- $\mathcal{N}(A)$ is all $\vec{x} \in R^n$ such that $A\vec{x} = \vec{0}_m$.

Question:

How can we interpret the row space?

Since the row space and the null space are both subspaces of R^n, we can ask how they are related. Let's remember that the product

$$A\vec{x} = \langle \text{Row}_1(A) \cdot \vec{x}, \text{Row}_2(A) \cdot \vec{x}, \ldots, \text{Row}_m(A) \cdot \vec{x} \rangle$$

Question from Exam 1

Let $\vec{u}$ and $\vec{v}$ be two vectors in R^n. Suppose $\vec{x}$ is a vector in R^n such that $\vec{x}$ is orthogonal to $\vec{u}$ and $\vec{x}$ is orthogonal to $\vec{v}$. Show that $\vec{x}$ is orthogonal to every vector in $\text{Span}\{\vec{u}, \vec{v}\}$.

This result generalizes. That is, if

$$\vec{x}\cdot\vec{v}_1 = 0, \quad \text{and} \quad \vec{x}\cdot\vec{v}_2 = 0, \quad \text{and} \quad \vec{x}\cdot\vec{v}_3 = 0, \quad \cdots, \quad \text{and} \quad \vec{x}\cdot\vec{v}_m = 0$$

then

$$\vec{x}\cdot\vec{z} = 0$$

for every vector $\vec{z}$ in $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_m\}$.

The Row Space

Suppose $\vec{x} \in \mathcal{N}(A)$ for some $m \times n$ matrix A. Then $A\vec{x} = \vec{0}_m$ which means that

$$\begin{aligned} \text{Row}_1(A) \cdot \vec{x} &= 0 \\ \text{Row}_2(A) \cdot \vec{x} &= 0 \\ \vdots \quad &\vdots \quad \vdots \\ \text{Row}_m(A) \cdot \vec{x} &= 0 \end{aligned}$$

That is, a vector $\vec{x} \in \mathcal{N}(A)$ is orthogonal to every row vector of A. Since that means that $\vec{x}$ is orthogonal to every linear combination of the row vectors of A, we can say

> Every vector in $\mathcal{RS}(A)$ is orthogonal to every vector in $\mathcal{N}(A)$ and vice versa.

Orthogonal Complements

Let W be a subspace of R^n. The **orthogonal complement** of W, denoted $W^\perp$, is the set of all $\vec{x}$ in R^n that are orthogonal to all vectors in W. We can write

$$W^\perp = \left\{\vec{x} \in R^n \mid \vec{x} \cdot \vec{w} = 0, \ \text{ for all } \ \vec{w} \in W\right\}.$$

The symbol $W^\perp$ is read *"W perp."*

$\mathcal{RS}(A)$ & $\mathcal{N}(A)$

For $m \times n$ matrix A, the row space of A is the orthogonal complement of the null space of A.

$$\mathcal{RS}(A) = \mathcal{N}(A)^\perp \quad \text{and} \quad \mathcal{N}(A) = \mathcal{RS}(A)^\perp.$$

$\mathcal{RS}(A)$ & $\mathcal{N}(A)$ of $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

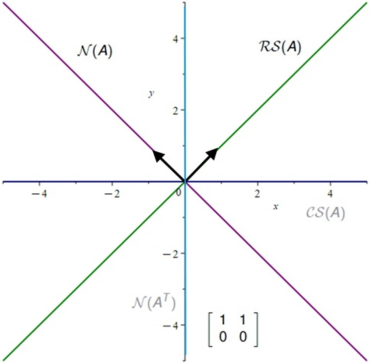

Figure: The row and null spaces of this matrix A are the lines $x_2 = x_1$ and $x_2 = -x_1$, respectively. In this case, they are actually perpendicular lines.

Orthogonal Complements in R^3

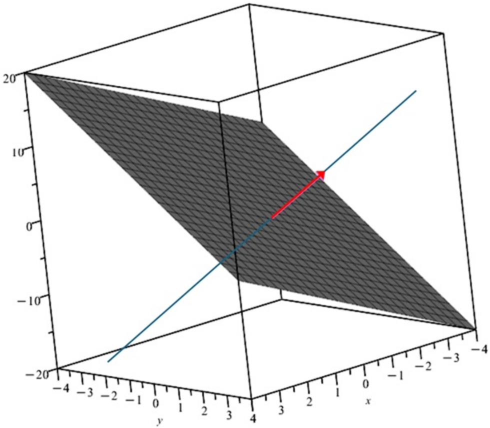

Figure: A subspace of R^3 that corresponds to a plane together with its orthogonal complement corresponding to a line.

The Fourth Fundamental Subspace

The fourth fundamental subspace of a matrix A is the null space of A^T, i.e., $\mathcal{N}(A^T)$. Recall that for a matrix A,

$$\text{Col}_i(A) = \text{Row}_i(A^T) \quad \text{and} \quad \text{Row}_i(A) = \text{Col}_i(A^T).$$

So this fourth subspace is the orthogonal complement of $\mathcal{CS}(A)$.

$\mathcal{N}(A^T)$

For $m \times n$ matrix A

$$\mathcal{N}(A^T) = \left\{ \vec{x} \in R^m \mid A^T\vec{x} = \vec{0}_n \right\}.$$

Equivalently

$$\mathcal{N}(A^T) = \{ \vec{x} \in R^m \mid \vec{x} \cdot \vec{y} = 0, \text{ for every } \vec{y} \in \mathcal{CS}(A) \}.$$

$\mathcal{CS}(A)$ & $\mathcal{N}(A^T)$ of $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

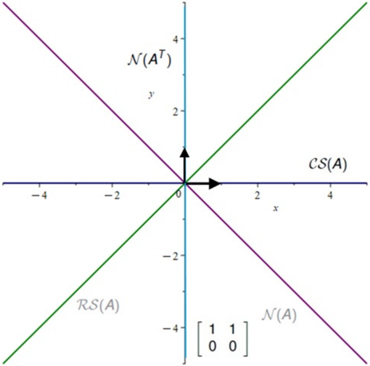

Figure: The column space of A and null space of A^T are the lines $x_2 = 0$ and $x_1 = 0$, respectively. These are also perpendicular line.

Example

Find a spanning set for each of the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 4 \\ 2 & -4 & 1 & -1 \end{bmatrix}.$$

4.3 Bases

Consider the two subspaces of R^2:

$$S_1 = \text{Span}\{\langle 1, 0\rangle\} \quad \text{and} \quad S_2 = \text{Span}\{\langle 1, 0\rangle, \langle 2, 0\rangle\}.$$

How are these related?

Bases

$$\text{Span}\{\langle 1,0\rangle\} = \text{Span}\{\langle 1,0\rangle, \langle 2,0\rangle\}$$

Note that $\{\langle 1,0\rangle\}$ is a linearly independent set and $\{\langle 1,0\rangle, \langle 2,0\rangle\}$ is a linearly dependent set. We might argue that $\{\langle 1,0\rangle\}$ is a more *efficient* spanning set.

Definition of a Basis

Let S be a subspace of R^n, and let $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ be a subset of vectors in S. $\mathcal{B}$ is a **basis** of S provided

- $\mathcal{B}$ spans S, and
- $\mathcal{B}$ is linearly independent.

A **basis** is a **linearly independent** spanning set. We can think of a basis as a minimal spanning set.

Standard a.k.a. Elementary Basis of R^n

The set $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of standard unit vectors in R^n is called the **standard basis** or the **elementary basis** of R^n.

For example,

$$R^2 = \mathrm{Span}\{\vec{e}_1, \vec{e}_2\},$$

$$R^3 = \mathrm{Span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\},$$

and so forth.

Elementary bases are easy to work with, but they're not the only bases we can work with.

Example

Show that $\{\langle 1, 1\rangle, \langle 1, -1\rangle\}$ is a basis for R^2.

Example

Determine whether the set $\{\langle 1, 0, 0\rangle, \langle 1, 1, 0\rangle, \langle 1, 1, 1\rangle, \langle 0, 1, 0\rangle\}$ is a basis for R^3.

Basis for a Null Space

Find a basis for $\mathcal{N}(A)$ for $A = \begin{bmatrix} -2 & -5 & 3 \\ 4 & 8 & 0 \\ -5 & -6 & -12 \end{bmatrix}$.

Why are Bases Special?

Coordinate Vectors

Theorem

Let $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ be an ordered basis of a subspace S of R^n. If $\vec{x}$ is any element of S, then there is exactly one representation (i.e., one set of coefficients) of $\vec{x}$ as a linear combination of elements of $\mathcal{B}$.

Note: Saying the basis is **ordered** just means that we put them in a particular order and number them accordingly.

Definition: Coordinate Vectors

Let S be a subspace of R^n and $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ be an ordered basis of S. For each element $\vec{x}$ in S, the **coordinate vector for $\vec{x}$ relative to the basis $\mathcal{B}$** is denoted $[\vec{x}]_{\mathcal{B}}$ and is defined to be

$$[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2, \ldots, c_k \rangle,$$

where the entries are the coefficients of the representation of $\vec{x}$ as a linear combination of the basis elements. That is, the c's are the coefficients in the equation

$$\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k.$$

Example

Consider the basis $\mathcal{B} = \{\langle 2, 1\rangle, \langle -1, 1\rangle\}$, in the order given, of R^2.
Determine

1. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle 2, 1\rangle$
2. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle -1, 1\rangle$
3. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle 1, 0\rangle$
4. $\vec{x}$ if $[\vec{x}]_{\mathcal{B}} = \langle -1, -1\rangle$

$\mathcal{B} = \{\langle 2, 1\rangle, \langle -1, 1\rangle\}$

$\mathcal{B} = \{\langle 2, 1\rangle, \langle -1, 1\rangle\}$

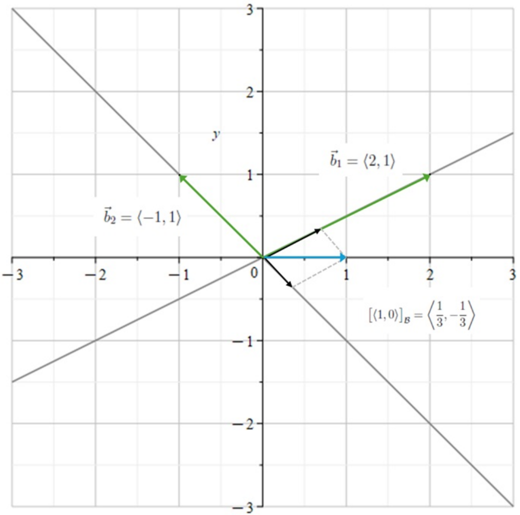

Figure: The coordinate vector $[\langle 1, 0\rangle]_{\mathcal{B}} = \left\langle \frac{1}{3}, -\frac{1}{3}\right\rangle$ because the vector we usually associate with $\vec{e}_1$ is $\frac{1}{3}\vec{b}_1 - \frac{1}{3}\vec{b}_2$ in the new basis $\mathcal{B} = \{\langle 2, 1\rangle, \langle -1, 1\rangle\}$.

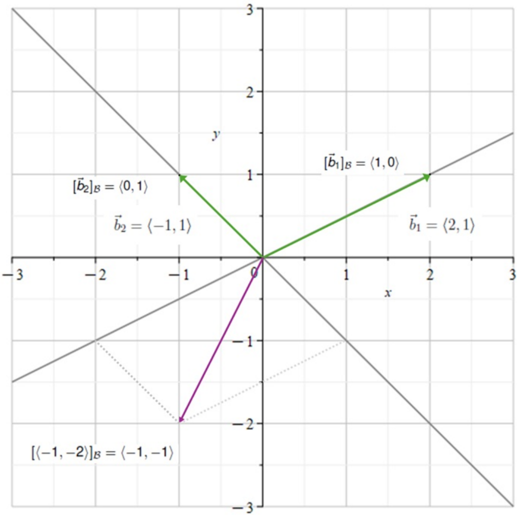

Figure: The coordinate vector $[\vec{x}]_{\mathcal{B}} = \langle -1, -1 \rangle$ is obtained by adding $-1\vec{b}_1$ and $-1\vec{b}_2$. In the standard coordinate system, this would correspond to the vector $\vec{x} = \langle -1, -2 \rangle$.

Change of Basis Matrix

Consider our example $\mathcal{B} = \{\langle 2,1\rangle, \langle -1,1\rangle\}$ for R^2. We can create a matrix B having the basis elements as its columns,

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

If $[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2\rangle$ for vector $\vec{x}$, then

$$\vec{x} = \underbrace{c_1\langle 2,1\rangle + c_2\langle -1,1\rangle}_{\text{lin. combo of columns}} = B[\vec{x}]_{\mathcal{B}}$$

B is called a **change of basis matrix.**

If B happens to be square, we can also get

$$[\vec{x}]_{\mathcal{B}} = B^{-1}\vec{x}.$$

(This is only relevant when the matrix is square.)

Example

Let $\mathcal{C} = \{\langle 1, 1, 0\rangle, \langle 0, 1, 0\rangle\}$ be an ordered basis for $S = \mathrm{Span}(\mathcal{C})$. Create a matrix C having the basis elements as its columns. Use the fact that $\vec{x} = C[\vec{x}]_{\mathcal{C}}$ to evaluate

1. $\vec{x}$ if $[\vec{x}]_{\mathcal{C}} = \langle 4, 2\rangle$
2. $[\vec{u}]_{\mathcal{C}}$ if $\vec{u} = \langle 2, -3, 0\rangle$

Isomorphic

We might notice that the subspace $S = \text{Span}\{\langle 1,1,0\rangle, \langle 0,1,0\rangle\}$ in the last example is a subspace of R^3. But we can equate each element uniquely with an element of R^2, namely its coordinate vector. Since we can equate the variable change to matrix multiplication the two operations, vector addition and scalar multiplication, are preserved when working with coordinate vectors. In fact, for every $\vec{u}$ and $\vec{v}$ in S and scalars c and d, it is true that

$$[c\vec{u} + d\vec{v}]_{\mathcal{C}} = c[\vec{u}]_{\mathcal{C}} + d[\vec{v}]_{\mathcal{C}}.$$

There is a name for this property.

We say that S is **isomorphic** to R^2.

4.3.2 Dimension

Theorem:

Suppose S is a subspace of R^n and $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a basis for S that contains k vectors with $k \geq 1$. If $T = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ is any set of m vectors in S with $m > k$, then T is linearly dependent.

Remark: This generalizes the result we had before that a set containing more vectors than elements in each vector must be linearly dependent. It says that a set of vectors having more vectors than elements in a basis for the subspace must be linearly dependent.

Example

1. If S has a basis $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ with three vectors in it, then **every set of vectors in S with four or more vectors in automatically linearly dependent.**

2. If P has a basis $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5\}$ with five vectors in it, then **every set of vectors in P with six or more vectors in automatically linearly dependent.**

Let $\mathcal{B} = \{\langle 1, 2, 0\rangle, \langle 0, 1, 1\rangle\}$ and $S = \mathrm{Span}(\mathcal{B})$. The subset of S,

$$\{\langle 0, 3, 3\rangle, \langle 1, 3, 1\rangle, \langle 2, 5, 1\rangle\}.$$

must be linearly dependent.

Dimension Defined

Theorem

Let $n \geq 2$ and $1 \leq k \leq n$. Suppose S is a subspace of R^n and $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ is a basis of S. Every basis of S consists of exactly k vectors.

Definition: Dimension

Let S be a subspace of R^n. If $S = \{\vec{0}_n\}$, then the dimension of S, written $\dim(S)$ is equal to zero. If S contains more than the zero vector, then the dimension of S, $\dim(S) = k$, where k is the number of elements in any basis of S.

The Dimension of R^n

Note that for $n \geq 2$, $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis for R^n. Hence

$$\dim(R^n) = n.$$

Example: What is the dimension of $\mathcal{N}(A)$ for
$A = \begin{bmatrix} -2 & -5 & 3 & -3 \\ 4 & 8 & 0 & 4 \\ -5 & -6 & -12 & -1 \end{bmatrix}$?

4.4 Bases for the Column & Row Spaces of a Matrix

Theorem

Let A be an $m \times n$ matrix that is not the zero matrix. Then the pivot columns of A form a basis for $\mathcal{CS}(A)$.

Theorem

If A and B are row equivalent matrices, then $\mathcal{RS}(A) = \mathcal{RS}(B)$.

Corollary

Let A be an $m \times n$ matrix that is not the zero matrix. Then the nonzero rows of $\mathrm{rref}(A)$ form a basis for $\mathcal{RS}(A)$.

Bases for Fundamental Subspaces

Given $m \times n$ matrix A that is not the zero matrix:

- Set up $\left[A \mid \vec{0}_m\right]$ and row reduce to $\left[\text{rref}(A) \mid \vec{0}_m\right]$.
- Identify the pivot columns from $\text{rref}(A)$ and use those pivot columns to form a basis for $\mathcal{CS}(A)$.
- Take the nonzero rows of $\text{rref}(A)$ to form a basis for $\mathcal{RS}(A)$.
- Use $\text{rref}(A)$ to deduce the relationship between basic and free variables, and use the factoring process to obtain a basis for $\mathcal{N}(A)$. If $\mathcal{N}(A) = \{\vec{0}_n\}$, then $\mathcal{N}(A)$ doesn't have a basis.
- If a basis for $\mathcal{N}(A^T)$ is desired, use $\left[\text{rref}(A^T) \mid \vec{0}_n\right]$ and the factoring process to obtain a basis.

Find Bases & Dimensions of $\mathcal{RS}(A)$, $\mathcal{CS}(A)$ and $\mathcal{N}(A)$

$$A = \begin{bmatrix} 2 & 6 & 0 & -2 & -4 \\ 1 & 3 & 1 & -4 & -17 \\ -1 & -3 & -1 & 4 & 17 \\ 2 & 6 & 1 & 0 & 1 \end{bmatrix}.$$

$$[A|\ \vec{0}_4] = \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Row Operations & Linear Dependence Relations

- Elementary row operations preserve linear dependence relations between columns but change the column space.
- Elementary row operations preserve the row space but change linear dependence relations between the rows.

Important Observations

- The basis elements for the column space **come from** A not from $\text{rref}(A)$.
- The basis elements for the row space **come from** $\text{rref}(A)$ not from A.
- The method we've been using all along to characterize solutions to $A\vec{x} = \vec{0}_m$ gives us a basis for the null space.

4.5 The Fundamental Theorem of Linear Algebra

Dimensions of $\mathcal{CS}(A)$, $\mathcal{RS}(A)$ & $\mathcal{N}(A)$

For $m \times n$ matrix A,

$$\dim(\mathcal{CS}(A)) = \text{ the number of pivot columns of } A.$$

$$\dim(\mathcal{N}(A)) = \text{ the number of non-pivot columns of } A.$$

$$\dim(\mathcal{RS}(A)) = \text{ the number of pivot columns of } A.$$

Rank & Nullity

Definition: Rank

The **rank** of a matrix A, denoted $\mathrm{rank}(A)$, is the dimension of the column space of A.

We also have a special name for the dimension of the null space of a matrix. We call this the nullity.

Definition: Nullity

The **nullity** of a matrix A, denoted $\mathrm{nullity}(A)$, is the dimension of the null space of A.

The Fundamental Theorem of Linear Algebra

Let A be an $m \times n$ matrix. Then

1. $\text{rank}(A) = \dim\big(\mathcal{CS}(A)\big) = \dim\big(\mathcal{RS}(A)\big)$.
2. $\text{rank}(A) + \text{nullity}(A) = n$.
3. Every vector $\vec{x}$ in $\mathcal{RS}(A)$ is orthogonal to every vector $\vec{y}$ in $\mathcal{N}(A)$, and similarly, every vector $\vec{u}$ in $\mathcal{CS}(A)$ is orthogonal to every vector $\vec{v}$ in $\mathcal{N}(A^T)$.

Part 2. of the FTLA is often called the *rank-nullity theorem*. It follows from the observation that

$$\begin{array}{rl} & \text{the number of pivot columns of } A \\ + & \text{the number of non-pivot columns of } A \\ \hline = & \text{the total number of columns of } A. \end{array}$$

Example

Suppose A is a 12×20 matrix.

1. If $\text{rank}(A) = 9$, how many free variables are there for $A\vec{x} = \vec{0}_{12}$?

2. If $\text{rref}(A)$ has seven nonzero rows, what is $\text{nullity}(A^T)$?

Example

Let $B = \begin{bmatrix} 1 & 3 & 1 & 0 & 2 \\ 2 & 2 & -2 & 4 & 0 \\ 3 & 1 & -5 & 8 & 1 \end{bmatrix}$. Find the rank and nullity of B.

$\text{rank}(B) =$ ______________ and $\text{nullity}(B) =$ ______________.