

Chapter 4 Vector Spaces & Subspaces

In this chapter, we will

- ▶ learn about additional properties of vectors in R^n ,
- ▶ learn about special subsets of R^n , including some related to matrices,
- ▶ state the **Fundamental Theorem of Linear Algebra**,
- ▶ and pin down precisely what a **vector space** is.

4.1 Linear Independence

Definition: Linear Independence

The collection of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ in R^m is said to be **linearly independent** if the homogeneous equation

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_n \vec{v}_n = \vec{0}_m \quad (1)$$

has only the trivial solution, $x_1 = x_2 = \dots = x_n = 0$.

If the collection of vectors is not linearly independent, then we say that it is **linearly dependent**.

For a linearly dependent set of vectors, an equation of the form (1) having at least one nonzero weight is called a **linear dependence relation**.

Some Observations on Linear (In)dependence

- ▶ Every nonempty set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in R^m is either linearly independent or linearly dependent.
- ▶ Linear independence/dependence is a property of a set (or collection) of vectors.

"The column vectors of A are linearly dependent." (makes sense)

"The matrix A is linearly dependent." (doesn't make sense)

- ▶ We saw last time that a set containing one vector, $\{\vec{v}\}$, in R^m is **linearly**

$$\begin{cases} \text{independent if } \vec{v} \neq \vec{0}_m \\ \text{dependent if } \vec{v} = \vec{0}_m \end{cases}$$

So $\{\langle 1, 0, 2, 1 \rangle\}$ is linearly independent, while $\{\langle 0, 0, 0 \rangle\}$ is linearly dependent.

The set $\{\vec{e}_1, \vec{e}_2\}$ in R^2 is linearly independent.

Consider the equation

$$x_1 \vec{e}_1 + x_2 \vec{e}_2 = \vec{0}_2$$

$$x_1 \langle 1, 0 \rangle + x_2 \langle 0, 1 \rangle = \langle 0, 0 \rangle$$

$$\langle x_1, 0 \rangle + \langle 0, x_2 \rangle = \langle 0, 0 \rangle$$

$$\langle x_1, x_2 \rangle = \langle 0, 0 \rangle$$

Comparing entries, $x_1 = 0$ and $x_2 = 0$.

$x_1 \vec{e}_1 + x_2 \vec{e}_2 = \vec{0}_2$ has only the trivial solution. So $\{\vec{e}_1, \vec{e}_2\}$ is linearly independent.

Show that the set $\{\langle 1, 0, 1 \rangle, \langle -3, 0, -3 \rangle\}$ is linearly dependent.

Consider $x_1 \langle 1, 0, 1 \rangle + x_2 \langle -3, 0, -3 \rangle = \langle 0, 0, 0 \rangle.$

$$\langle x_1 - 3x_2, 0, x_1 - 3x_2 \rangle = \langle 0, 0, 0 \rangle.$$

This is true for any pair x_1, x_2 such that $x_1 - 3x_2 = 0$, i.e., $x_1 = 3x_2$.

An example of a linear dependence relation is

$$3 \langle 1, 0, 1 \rangle + 1 \langle -3, 0, -3 \rangle = \langle 0, 0, 0 \rangle.$$

So $\{ \langle 1, 0, 1 \rangle, \langle -3, 0, -3 \rangle \}$ is
linearly dependent.

A Set of Two Vectors

A set of two vectors, $\{\vec{v}_1, \vec{v}_2\}$, in R^n is linearly dependent if and only if one of the vectors is a scalar multiple of the other.

Let's show that if one vector is a scalar multiple of the other, the set is linearly dependent.

Suppose $\vec{v}_1 = c\vec{v}_2$ for some scalar c .

This rearranges to $\vec{v}_1 - c\vec{v}_2 = \vec{0}_n$.

The coefficient of $\vec{v}_1$ is $1 \neq 0$. This is a linear dependence relation. So $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent.

Now, let's show that if $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent, then one vector is a scalar multiple of the other.

Since $\{\vec{v}_1, \vec{v}_2\}$ is lin. dependent, there exist scalars x_1, x_2 , not both zero,

such that $x_1 \vec{v}_1 + x_2 \vec{v}_2 = \vec{0}_n$. Assuming

$x_1 \neq 0$, rearrange to get

$$x_1 \vec{v}_1 = -x_2 \vec{v}_2 \Rightarrow$$

$$\vec{v}_1 = \frac{-x_2}{x_1} \vec{v}_2.$$

Thus $\vec{v}_1$ is a scalar multiple of $\vec{v}_2$.

Example

Identify each set as being linearly dependent or linearly independent.

1. $\{\langle 1, 2, 1 \rangle\}$ Lin. Independent.

2. $\{\langle 4, 2, -1, 0 \rangle, \langle -8, -4, 2, 0 \rangle\}$ Lin. dependent

3. $\{\langle 1, 1 \rangle, \langle 0, 0 \rangle\}$ Lin. dependent $\langle 0, 0 \rangle = 0 \langle 1, 1 \rangle$.

4. $\{\langle 1, 3, 0, 4 \rangle, \langle 2, 0, 6, 8 \rangle\}$ Lin. Independent.

Three or More Vectors

With a set of three or more vectors, we can always turn an equation like

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n = \vec{0}_m$$

into a matrix-vector equation

$$A\vec{x} = \vec{0}_m$$

by setting

$$\text{Col}_i(A) = \vec{v}_i, \quad i = 1, \dots, n.$$

Example: Determine whether the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent or linearly independent where

$$\vec{v}_1 = \langle -2, 4, -5 \rangle, \quad \vec{v}_2 = \langle -5, 8, -6 \rangle, \quad \vec{v}_3 = \langle 3, 0, -12 \rangle.$$

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Let A be a matrix with columns $\text{Col}_i(A) = \vec{v}_i$

for $A\vec{x} = \vec{0}_3$,

$$[A \mid \vec{0}_3] = \left[\begin{array}{ccc|c} -2 & -5 & 3 & 0 \\ 4 & 8 & 0 & 0 \\ -5 & -6 & -12 & 0 \end{array} \right] \xrightarrow{\text{rref}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$A\vec{x} = \vec{0}_3$ has non trivial solutions. Hence $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is

linearly dependent.

$$\vec{v}_1 = \langle -2, 4, -5 \rangle, \quad \vec{v}_2 = \langle -5, 8, -6 \rangle, \quad \vec{v}_3 = \langle 3, 0, -12 \rangle$$

$$\text{Claim} \quad \vec{v}_3 = 6\vec{v}_1 + (-3)\vec{v}_2$$

$$\begin{aligned} 6\vec{v}_1 - 3\vec{v}_2 &= 6\langle -2, 4, -5 \rangle - 3\langle -5, 8, -6 \rangle \\ &= \langle -12, 24, -30 \rangle + \langle 15, -24, 18 \rangle \\ &= \langle 3, 0, -12 \rangle \end{aligned}$$

Part of what this means is that we can use the rref to write a linear dependence relation for $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{v}_3 = 6\vec{v}_1 - 3\vec{v}_2 \Rightarrow$$

$$6\vec{v}_1 - 3\vec{v}_2 - \vec{v}_3 = \vec{0}_3$$