October 11 Math 2306 sec. 51 Fall 2021

Section 11: Linear Mechanical Equations

The displacement $x(t)$ at the time t of an object subjected to a spring force and damping force satisfied the ODE

$$mx'' + \beta x' + kx = 0.$$

- m is the mass,
- β is the damping coefficient, and
- k is the spring constant.

This was derived by summing the forces. Total force $F = mx''$, damping force $F_{\text{damping}} = \beta x'$, spring force $F_{\text{spring}} = kx$

Free Damped Motion

The equation in standard form is

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

Damping Types $\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$

The roots[1] of the characteristic equation are

$$r = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

The motion is called

- **over damped** if there are two, distinct real roots (decay only, no oscillations)
- **critically damped** if there is one, repeated real root (fasted decay, no oscillations), and
- **under damped** if the roots are complex conjugates (decay with oscillations).

[1] **Observation:** Conservation of energy ensures that for all cases with damping, the **real** part of the roots of the characteristic equation ($-\lambda$) MUST be negative.

Comparison of Damping

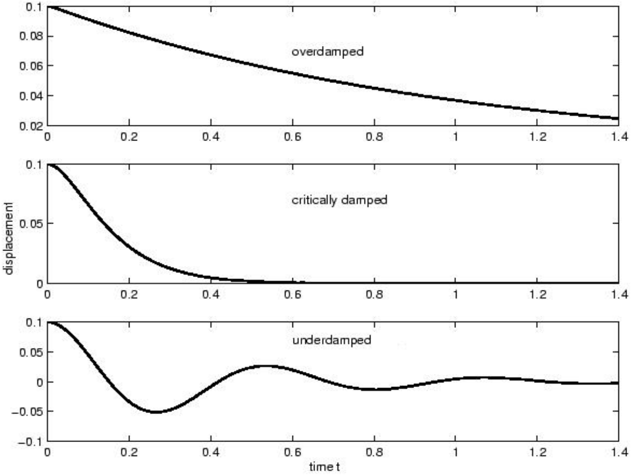

Figure: Comparison of motion for the three damping types.

Initial Conditions

Given an initial position $x(0) = x_0$ and initial velocity $x'(0) = x_1$, the displacement will satisfy an initial value problem

$$mx'' + \beta x' + kx = 0 \quad x(0) = x_0 \quad x'(0) = x_1$$

A couple of terms: If an object is released

- **from equilibrium**, it means that $x(0) = 0$;
- **from rest**, it means that $x'(0) = 0$.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The ODE is $mx'' + \beta x' + kx = 0$

$m = 2$ kg

$k = 12$ N/m

$\beta = 10$

$2x'' + 10x' + 12x = 0$

In standard form

$x'' + 5x' + 6x = 0$

The characteristic eqn is

$$r^2 + 5r + 6 = 0$$

$$(r+3)(r+2) = 0$$

$$r = -3 \quad \text{or} \quad r = -2$$

Two distinct real roots $\Rightarrow$

the system is

$\underline{\underline{\text{overdamped}}}$ this is the answer

Aside: $x'' + 5x' + 6x = 0$

$$2\lambda = \frac{\beta}{m} = 5 \implies \lambda = \frac{5}{2}$$

$$\omega^2 = \frac{k}{m} = 6$$

Note $\lambda^2 - \omega^2 = \left(\frac{5}{2}\right)^2 - 6$

$$= \frac{25}{4} - \frac{24}{4} = \frac{1}{4} > 0$$

Again, we'd conclude that the system is over damped

Example

A 64 lb object stretches a spring 4 ft in equilibrium. It is attached to a dashpot with damping constant $\beta = 8$ lb-sec/ft. The object is initially displaced 18 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement for all $t > 0$.

The ODE is $mx'' + \beta x' + kx = 0$

We're given $\beta = 8$, we need m and k.

The weight $W = 64$ lb. The mass

$$m = \frac{W}{g} = \frac{64\text{ lb}}{32\text{ ft/sec}^2} = 2 \text{ slugs}$$

The spring constant

$$k = \frac{W}{\delta x} = \frac{64\text{ lb}}{4\text{ ft}} = 16\ \frac{\text{lb}}{\text{ft}}$$

The ODE is $2x'' + 8x' + 16x = 0$

The IC are $x(0) = 1.5$ ft

$x'(0) = -4$ ft/sec

In standard form the ODE is

$$x'' + 4x' + 8x = 0$$

w/ characteristic eqn

$$r^2 + 4r + 8 = 0$$

$$r^2 + 4r + 4 + 4 = 0$$

$$(r+2)^2 = -4$$

$$r + 2 = \pm\sqrt{-4} = \pm 2i$$

$r = -2 \pm 2i$ underdamped

The solutions

$$x_1 = e^{-2t}\cos(2t), \quad x_2 = e^{-2t}\sin(2t)$$

$$x = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t)$$

This is the general solution.

Apply $x(0) = 1.5$, $x'(0) = -4$

$$x' = -2c_1 e^{-2t}\cos(2t) - 2c_1 e^{-2t}\sin(2t) - 2c_2 e^{-2t}\sin(2t) + 2c_2 e^{-2t}\cos(2t)$$

$$e^0 = 1 \quad \cos(0) = 1 \quad \sin(0) = 0$$

$$x(0) = c_1 = 1.5 \Rightarrow c_1 = 1.5$$

$$x'(0) = -2c_1 + 2c_2 = -4$$

$$c_2 = -2 + c_1 = -2 + 1.5 = -0.5$$

The position for all $t > 0$ is

$$\boxed{x = 1.5e^{-2t}\cos(2t) - 0.5e^{-2t}\sin(2t)}$$

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and} \qquad (2) \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$$x_p = A\cos(\gamma t) + B\sin(\gamma t)$$

If $\gamma \neq \omega$ then

x_p has no like terms in common w/ x_c. This guess is correct and

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + A\cos(\gamma t) + B\sin(\gamma t)$$

$x'' + \omega^2 x = F_0 \sin(\gamma t)$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ If $\gamma = \omega$, this guess won't work (like terms in common w/ x_c).

$$x_p = (A\cos(\gamma t) + B\sin(\gamma t))t$$
$$= At\cos(\omega t) + Bt\sin(\omega t)$$

$$x = c_1\cos(\omega t) + c_2\sin(\omega t) + At\cos(\omega t) + Bt\sin(\omega t)$$

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2} \left(\sin(\gamma t) - \frac{\gamma}{\omega} \sin(\omega t) \right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

$$\text{Case (2):} \quad x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{2\omega^2} \sin(\omega t) - \frac{F_0}{2\omega} t \cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

▸ Forced Motion and Resonance Applet

Choose ”Elongation diagram” to see a plot of displacement. Try exciter frequencies close to ω.