October 11 Math 2306 sec. 52 Spring 2023

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

The general solution, $y = y_c + y_p$ will require both y_c and y_p. The associated homogeneous equation will be constant coefficient, so we use the method of the last section to find y_c.

Method Basics: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$

- Classify g as a certain *type*, and assume y_p is of this same type[1] with unspecified coefficients, A, B, C, etc.
- Substitute the assumed y_p into the ODE and collect like terms
- Match like terms on the left and right to get equations for the coefficients.
- Solve the resulting system to determine the coefficients for y_p.

[1] We will see shortly that our final conclusion on the format of y_p can depend on y_c.

Some Rules & Caveats

Rules of Thumb

- Polynomials include all powers from constant up to the degree.
- Where sines go, cosines follow and vice versa.

Caution

- The method is self correcting, but it's best to get the set up correct.
- If an initial *guess* for y_p shares like term(s) in common with y_c, include a factor x^n, where n is the smallest positive integer needed to eliminate any common like term.

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4$$

Find y_c first: Characteristic eqn

$m = \alpha \pm \beta i$

$$m^3 - m^2 + m - 1 = 0$$

$$m^2(m-1) + (m-1) = 0$$

$$(m^2+1)(m-1) = 0 \Rightarrow$$

$m - 1 = 0 \quad m = 1$

$m^2 + 1 = 0 \quad m = \pm i$

$\alpha = 0 \quad \beta = 1$

$$y_1 = e^{1x}, \quad y_2 = e^{0x}\cos x, \quad y_3 = e^{0x}\sin x$$

$$y_c = c_1 e^x + c_2 \cos x + c_3 \sin x$$

Let's look for y_p in the form $y_p = y_{p_1} + y_{p_2}$

where y_{p_1} solves

$$y_{p_1}''' - y_{p_1}'' + y_{p_1}' - y_{p_1} = \cos x$$

and y_{p_2} solves

$$y_{p_2}''' - y_{p_2}'' + y_{p_2}' - y_{p_2} = x^4$$

$$y_c = c_1 e^x + c_2 \cos x + c_3 \sin x$$

For $g_1(x) = \cos x$
$$y_{p_1} = (A \cos x + B \sin x)x$$
$$y_{p_1} = Ax \cos x + Bx \sin x \quad \checkmark$$

For $g_2(x) = x^4$
$$y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G \quad \checkmark$$

Then $y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2y'' + xy' - 4y = e^x.$$

Question: Can the method of undetermined coefficients be used to find a particular solution for either of these nonhomogeneous ODEs? (Why/why not?)

tan x is not an allowable right side,

the other is not constant coef.

Variation of Parameters

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x) \tag{1}$$

For the equation (1) in standard form suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

$y_c = c_1y_1(x) + c_2y_2(x)$ c_1, c_2 are constants

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We have two unknowns u_1 and u_2 and only one equation. We will introduce a 2nd equation.

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$$

$$\boxed{\text{Assume} \quad u_1' y_1 + u_2' y_2 = 0}$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

$$y_p'' + P(x) y_p' + Q(x) y_p = g(x)$$

$$u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + P(x)\left(u_1 y_1' + u_2 y_2'\right) + Q(x)\left(u_1 y_1 + u_2 y_2\right) = g(x)$$

Collect u_1', u_2', u_1, u_2

$$u_1' y_1' + u_2' y_2' + \underbrace{\left(y_1'' + P(x) y_1' + Q(x) y_1\right)}_{=0} u_1 + \underbrace{\left(y_2'' + P(x) y_2' + Q(x) y_2\right)}_{=0} u_2 = g(x)$$

$$\Rightarrow \boxed{u_1' y_1' + u_2' y_2' = g}$$

So u_1 and u_2 solve the system

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g$$

In matrix format

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

We'll finish finding u_1, u_2 on Friday.