October 11 Math 2306 sec. 53 Fall 2024

Section 9: Method of Undetermined Coefficients

We are considering nonhomogeneous, linear ODEs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

with restrictions on the left and right sides.

- The left side must be **constant coefficient.**
- The right side, $g(x)$, has to be one of the following function types:
  - polynomials,
  - exponentials,
  - sines and/or cosines,
  - and products and sums of the above kinds of functions

The **general solution** will have the form $y = y_c + y_p$. The process here is for finding y_p.

The Method of Undetermined Coefficients

1. Confirm the ODE has the right properties (constant coef. left, correct type[a] of right side).

2. Find y_c and classify the right hand side g.

3. Set up the guess y_p based on g.

4. Compare the y_p guess to y_c.
   - 4.1. If they have no like terms in common, proceed to the next step.
   - 4.2. If the y_p has one or more like terms in common with y_c, multiply by x enough times to eliminate all duplication.

5. Substitute the assumed y_p into the ODE and match *like terms* to find the coefficients that work.

[a] polynomial, exponential, sine/cosine, sum/product of these three

Examples

Find the general solution of the ODE.

$$y'' - 2y' + y = -4e^x$$

The left is constant coeff and the right side, $g(x) = -4e^x$, is an exponential. Find y_c. y_c solves $y'' - 2y' + y = 0$. The characteristic equation is $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$

$r = 1$ is a double root.

$y_1 = e^{1x}$, $y_2 = xe^{1x}$. So

$y_c = c_1 e^x + c_2 x e^x$

$$y'' - 2y' + y = -4e^x$$

Next, find y_p. $g(x) = -4e^x$, $y_p = Ae^x$.

This duplicates y_1 from y_c. Modifying y_p..

$y_p = (Ae^x)x^2 = Ax^2e^x$. This is correct.

Sub y_p into the ODE.

$$y_p = Ax^2e^x$$

$$y_p' = Ax^2e^x + 2Axe^x$$

$$y_p'' = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$$

$$= Ax^2e^x + 4Axe^x + 2Ae^x$$

$$y'' - 2y' + y = -4e^x$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = -4e^x$$

Collect x^2e^x, xe^x and e^x

$$x^2e^x\underbrace{(A-2A+A)}_{=0} + xe^x\underbrace{(4A-4A)}_{=0} + e^x(2A) = -4e^x$$

$$2Ae^x = -4e^x$$

$$2A = -4, \quad A = -2$$

So $y_p = -2x^2e^x$

with $y_c = c_1e^x + c_2xe^x$

The general solution, $y = y_c + y_p$,

$$y = c_1 e^x + c_2 x e^x - 2x^2 e^x$$

Find the form of the particular soluition

$$y'' + 6y' + 13y = xe^{-3x} + \cos(2x) + 4e^{-3x}\sin(2x)$$

Find y_c sinc it can affect y_p. y_c solves

$y'' + 6y' + 13y = 0$. The characteristic eqn is:

$$r^2 + 6r + 13 = 0 \Rightarrow r^2 + 6r + 9 = -4$$

$$(r+3)^2 = -4$$

$$r + 3 = \pm 2i \Rightarrow r = -3 \pm 2i$$

$\alpha \pm i\beta$, $-3 \pm 2i \Rightarrow \alpha = -3$ and $\beta = 2$

$$y_1 = e^{-3x}\cos(2x), \quad y_2 = e^{-3x}\sin(2x)$$

$$y'' + 6y' + 13y = xe^{-3x} + \cos(2x) + 4e^{-3x}\sin(2x)$$

We'll use superposition. Let y_{p_1} solve

$$y'' + 6y' + 13y = xe^{-3x} \; ; \quad g_1(x) = xe^{-3x}$$

Based on g_1, $y_{p_1} = (Ax+B)e^{-3x} = Axe^{-3x} + Be^{-3x}$

This is correct. Let y_{p_2} solve

$$y'' + 6y' + 13y = \cos(2x) \; ; \quad g_2(x) = \cos(2x).$$

Based on g_2, $y_{p_2} = C\cos(2x) + D\sin(2x)$

This is the correct y_{p_2} form.

$$y_1 = e^{-3x}\cos(2x), \quad y_2 = e^{-3x}\sin(2x)$$

$$y'' + 6y' + 13y = xe^{-3x} + \cos(2x) + 4e^{-3x}\sin(2x)$$

Let y_{p_3} solve $y'' + 6y' + 13y = 4e^{-3x}\sin(2x)$.

$g_3(x) = 4e^{-3x}\sin(2x)$, $y_{p_3} = Ee^{-3x}\sin(2x) + Fe^{-3x}\cos(2x)$

This duplicates y_c. Modify y_{p_3}:

$$y_{p_3} = \left(Ee^{-3x}\sin(2x) + Fe^{-3x}\cos(2x)\right)x$$

$$= Exe^{-3x}\sin(2x) + Fxe^{-3x}\cos(2x)$$

This is correct.

For the whole ODE,

$$\boxed{y_p = (Ax+B)e^{-3x} + C\cos(2x) + D\sin(2x) + Exe^{-3x}\sin(2x) + Fxe^{-3x}\cos(2x)}$$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4$$

Find y_c. y_c solves $y''' - y'' + y' - y = 0$.

The characteristic eqn is

$$r^3 - r^2 + r - 1 = 0$$

$$r^2(r-1) + (r-1) = 0 \Rightarrow (r^2+1)(r-1) = 0$$

$r = 1$ or $r = \pm i = 0 \pm 1i$ $\alpha = 0$, $\beta = 1$

$y_1 = e^{1x}$, $y_2 = e^{0x}\cos(1x)$, $y_3 = e^{0x}\sin(1x)$

$y_1 = e^x$, $y_2 = \cos x$, $y_3 = \sin x$

$$y''' - y'' + y' - y = \cos x + x^4$$

Let y_{p_1} solve, $y''' - y'' + y' - y = \cos x$

$g_1(x) = \cos x$, $y_{p_1} = (A\cos x + B\sin x)x$

$y_{p_1} = Ax\cos x + Bx\sin x$

Let y_{p_2} solve $y''' - y'' + y' - y = x^4$

$g_2(x) = x^4$, $y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$

For the whole ODE

$y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$