October 12 Math 2306 sec. 51 Fall 2022

Section 11: Linear Mechanical Equations

Simple Harmonic Motion

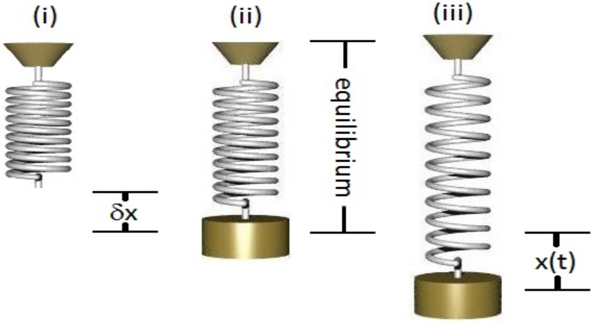

Figure: We track the displacement x from the equilibrium position. Recall δx is the displacement **in** equilibrium. We assume there are no damping nor external forces (for now).

Simple Harmonic Motion

We derive the equation for displacement. Given initial displacement x_0 and initial velocity x_1,

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

where $\omega^2 = \dfrac{k}{m} = \dfrac{g}{\delta x}$.

k-spring constant, m-object's mass, g-acceleration due to gravity.

Simple Harmonic Motion

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[1]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[1] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

We can formulate the solution in terms of a single sine (or cosine) function. Letting $A = \sqrt{x_0^2 + (x_1/\omega)^2}$,

$$x(t) = A \sin(\omega t + \phi) \quad \text{where}$$

$$\sin\phi = \frac{x_0}{A}, \quad \text{and} \quad \cos\phi = \frac{x_1}{\omega A}.$$

Or

$$x(t) = A \cos(\omega t - \hat{\phi}) \quad \text{where}$$

$$\cos\hat{\phi} = \frac{x_0}{A}, \quad \text{and} \quad \sin\hat{\phi} = \frac{x_1}{\omega A}.$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. After how many seconds is the mass at the equilibrium position for the first time? (Take $g = 32$ ft/sec^2.)

The ODE is $mx'' + kx = 0$. We can compute m and k.

$W = mg$ $\qquad\qquad$ $W = k\,\delta x$

$4\text{ lb} = m\left(32\,\frac{\text{ft}}{\text{sec}^2}\right)$, $\qquad$ $4\text{ lb} = k\,(6\text{ in}) = k\left(\frac{1}{2}\text{ ft}\right)$

$\Rightarrow m = \frac{4}{32}$ slugs $\qquad$ $k = \frac{4}{1/2}\,\frac{\text{lb}}{\text{ft}} = 8\,\frac{\text{lb}}{\text{ft}}$

$= \frac{1}{8}$ slugs

$$\omega^2 = \frac{k}{m} = \frac{8}{1/8} = 64$$

$\boxed{\text{The ODE is } x'' + 64x = 0}$

Note: $\omega^2 = \frac{g}{\delta x} = \frac{32\ ft/sec^2}{1/2\ ft} = 64\ \frac{1}{sec^2}$

The initial conditions are

$$\boxed{x(0) = 4, \quad x'(0) = -24}$$

The characteristic equation is (using r)

$$r^2 + 64 = 0 \implies r^2 = -64$$

$$r = \pm\sqrt{-64} = \pm 8i$$

$$x_1 = \cos(8t) \quad , \quad x_2 = \sin(8t)$$

$$x(t) = c_1 \cos(8t) + c_2 \sin(8t)$$

Apply $x(0) = 4$, $x'(0) = -24$

$$x(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 = 4$$

$$x'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t)$$

$$x'(0) = -8c_1 \sin(0) + 8c_2 \cos(0) = -24$$

$$8c_2 = -24 \Rightarrow c_2 = \frac{-24}{8} = -3$$

The position

$$\boxed{x(t) = 4\cos(8t) - 3\sin(8t)}$$

The period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

frequency $f = \frac{1}{T} = \frac{4}{\pi}$

Amplitude $A = \sqrt{4^2 + (-3)^2} = 5$

Expressing x in terms of a Sine function.

$$x(t) = A\sin(\omega t + \phi)$$

$$x(t) = 5\sin(8t + \phi)$$

where $\sin\phi = \frac{x_0}{A} = \frac{4}{5}$, $\cos\phi = \frac{x_1}{\omega A} = \frac{-3}{5}$

$$\phi = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21 \qquad \approx 127^\circ$$

The object is at the equilibrium position if $8t + \phi = n\pi$ for n an integer

The first time this happens is when

$n = 1$ $\quad 8t + \phi = \pi \Rightarrow 8t = \pi - \phi$

$$t = \frac{\pi - \phi}{8} \approx \frac{\pi - 2.21}{8} \approx 0.116$$

the first time is when

$t \approx 0.116$ sec.

Free Damped Motion

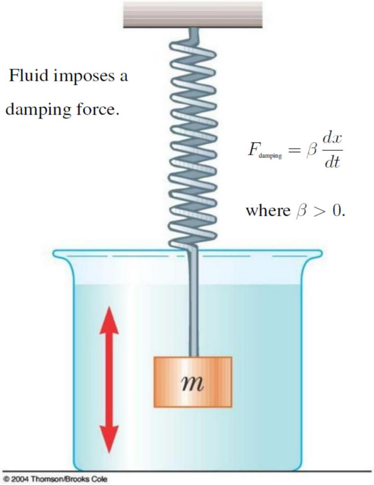

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of damping + Force of spring

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \qquad \Longrightarrow$$

$$mx'' + \beta x' + kx = 0$$

2nd order, linear, constant coefficient, homogeneous ODE.

$$x'' + \frac{\beta}{m}x' + \frac{k}{m}x = 0$$

Let $\omega^2 = \dfrac{k}{m}$

$2\lambda = \beta/m$

Free Damped Motion

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

This is a second order, linear, constant coefficient, homogeneous ODE with characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0$$

having roots

$$r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation.

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

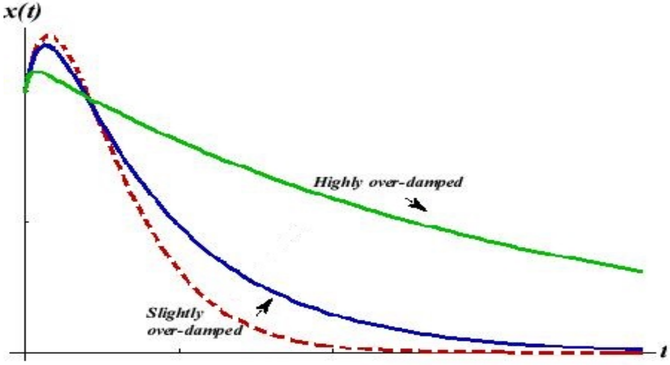

Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

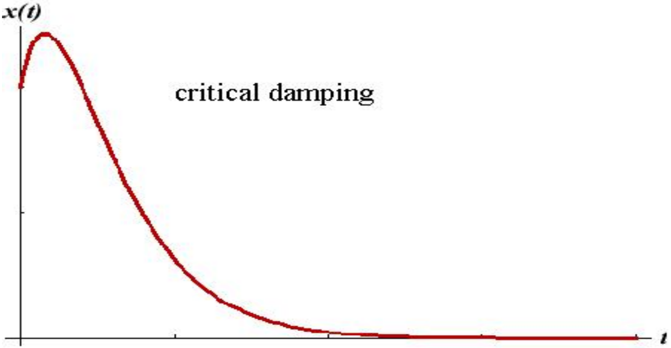

Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

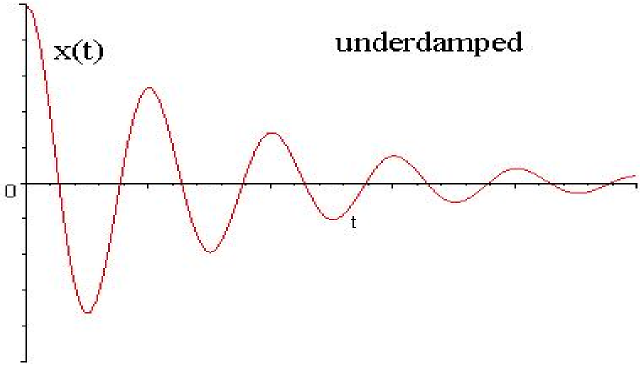

Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.