October 13 Math 2306 sec. 51 Spring 2023

Section 10: Variation of Parameters

We're considering a second order, nonhomogeneous linear ODE in standard form.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x) \tag{1}$$

Let $\{y_1(x),\ y_2(x)\}$ Be a fundamental solution set for the associated homogeneous equation.

Variation of parameters is a method for finding a particular solution y_p assuming that it can be found in the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

for some functions u_1 and u_2.

$$y'' + P(x)y' + Q(x)y = g(x)$$

We were in the process of deriving formulas for the functions u_1 and u_2 so that

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

We had arrived at the system of equations

$$\begin{array}{rcrcl} y_1 u_1' & + & y_2 u_2' & = & 0 \\ y_1' u_1' & + & y_2' u_2' & = & g, \end{array}$$

which can be stated using a matrix formalism as

$$\left[\begin{array}{cc} y_1 & y_2 \\ y_1' & y_2' \end{array}\right] \left[\begin{array}{c} u_1' \\ u_2' \end{array}\right] = \left[\begin{array}{c} 0 \\ g \end{array}\right].$$

Any method for solving linear systems can be used here, but we'll use Cramer's rule to solve for u_1' and u_2'. We integrate these to get u_1 and u_2.

Cramer's Rule

Consider the linear system of two equations in two unknowns

$$\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array} \quad \text{i.e.,} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

and define the three matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A_x = \begin{bmatrix} e & b \\ f & d \end{bmatrix}, \quad \text{and} \quad A_y = \begin{bmatrix} a & e \\ c & f \end{bmatrix}.$$

If $\det(A) \neq 0$, then the system is uniquely solvable[a], and the solution

$$x = \frac{\det(A_x)}{\det(A)} \quad \text{and} \quad y = \frac{\det(A_y)}{\det(A)}.$$

[a] This is a well known result that can be found in any elementary discussion of Linear Algebra.

Complete the Derivation of y_p

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

If $W = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$, let

$W_1 = \begin{bmatrix} 0 & y_2 \\ g & y_2' \end{bmatrix}$, $W_2 = \begin{bmatrix} y_1 & 0 \\ y_1' & g \end{bmatrix}$

$$u_1' = \frac{\det(W_1)}{\det(W)} = \frac{0 - gy_2}{\det(W)} = \frac{-gy_2}{W}$$

the determinant

$$u_2' = \frac{\det(W_2)}{\det(W)} = \frac{y_1 g - 0}{\det(W)} = \frac{gy_1}{W}$$

So $\quad u_1 = \int \frac{-g y_2}{W}\, dx, \quad u_2 = \int \frac{g y_1}{W}\, dx$

Variation of Parameters

$$y'' + P(x)y' + Q(x)y = g(x)$$

If $\{y_1, y_2\}$ is a fundamental solution set for the associated homogeneous equation, then the general solution is

$$y = y_c + y_p \quad \text{where}$$

$$y_c = c_1 y_1(x) + c_2 y_2(x), \quad \text{and} \quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Letting W denote the Wronskian of y_1 and y_2, the functions u_1 and u_2 are given by the formulas

$$u_1 = \int \frac{-y_2 g}{W}\,dx, \quad \text{and} \quad u_2 = \int \frac{y_1 g}{W}\,dx.$$

Example:

Find the general solution of the ODE $y'' + y = \tan x$.

Find y_c: y_c solves $y_c'' + y_c = 0$

Characteristic eqn $m^2 + 1 = 0 \Rightarrow m^2 = -1$

$m = \pm\sqrt{-1} = \pm i$ $\quad \alpha \pm \beta i$ where $\alpha = 0$, $\beta = 1$

$y_1 = e^{0x}\cos(1x)$, $y_2 = e^{0x}\sin(1x)$

$y_1 = \cos(x)$, $y_2 = \sin(x)$

$$\boxed{y_c = c_1 \cos x + c_2 \sin x}$$

Find y_p as $y_p = u_1 y_1 + u_2 y_2$

Here, $g(x) = \tan x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = 1$$

$y_1 = \cos x$, $y_2 = \sin x$, $g(x) = \tan x$ and $W = 1$

$$u_1 = \int \frac{-g y_2}{W}\,dx = \int -\frac{\tan x \sin x}{1}\,dx = -\int \frac{\sin x}{\cos x} \sin x\,dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{(1-\cos^2 x)}{\cos x}\,dx = \int \frac{\cos^2 x - 1}{\cos x}\,dx$$

$$= \int (\cos x - \sec x)\,dx$$

$$u_1 = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \frac{g y_1}{W}\,dx = \int \frac{\tan x \cos x}{1}\,dx$$

$$= \int \frac{\sin x}{\cos x} \cos x \, dx = \int \sin x \, dx$$

$$u_2 = -\cos x$$

$$y_1 = \cos x, \quad y_2 = \sin x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\sin x - \ln|\sec x + \tan x|\right)\cos x + (-\cos x)\sin x$$

$$= \sin x \cancel{\cos x} - \cos x \ln|\sec x + \tan x| - \cancel{\cos x \sin x}$$

$$= -\cos x \ln|\sec x + \tan x|$$

The general solution

$$\boxed{y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|}$$

This solves $y'' + y = \tan x$

Solve the IVP

$$x^2y'' + xy' - 4y = 8x^2, \quad y(1) = 1, \quad y'(1) = 1$$

The complementary solution of the ODE is $y_c = c_1x^2 + c_2x^{-2}$.

We need to find y_p. The ODE in standard form

is $\quad y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 8 \quad \Rightarrow \quad g(x) = 8$

$y_1 = x^2 \qquad y_2 = x^{-2}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = x^2(-2x^{-3}) - 2x(x^{-2})$$

$$= -2x^{-1} - 2x^{-1} = -4x^{-1}$$

$y_1 = x^2, \quad y_2 = x^{-2}, \quad W = -4x^{-1}, \quad g(x) = 8$

$y_p = u_1 y_1 + u_2 y_2$ where

$$u_1 = \int \frac{-g y_2}{W}\, dx, \qquad u_2 = \int \frac{g y_1}{W}\, dx$$

$$u_1 = \int \frac{-8x^{-2}}{-4x^{-1}}\, dx = 2\int x^{-2}\cdot x\, dx = 2\int x^{-1}\, dx = 2\ln|x|$$

$$u_2 = \int \frac{8x^2}{-4x^{-1}}\, dx = -2\int x^2\cdot x\, dx = -2\int x^3\, dx = -2\frac{x^4}{4}$$

$$u_1 = 2\ln x, \qquad u_2 = \frac{-1}{2}x^4$$

$$y_p = u_1 y_1 + u_2 y_2 = 2\ln x\,(x^2) + \left(\frac{-1}{2}x^4\right)(x^{-2})$$

$$y_p = 2x^2 \ln x - \frac{1}{2}x^2$$

The general solution $y_c + y_p$

$$y = C_1 x^2 + C_2 x^{-2} + 2x^2 \ln x - \frac{1}{2}x^2$$

$$y = k_1 x^2 + C_2 x^{-2} + 2x^2 \ln x$$ where $k_1 = C_1 - \frac{1}{2}$

We ran out of time and couldn't finish the IVP, but this is the general solution to the ODE.