October 13 Math 2306 sec. 52 Fall 2021

Section 11: Linear Mechanical Equations

Driven Motion: We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

We considered the case that the forcing was a simple sine (or cosine).

$$x'' + \omega^2 x = F_0 \sin(\gamma t).$$

The complementary solution is

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t),$$

and the particular solution will look like

$$x_p = A\cos(\gamma t) + B\sin(\gamma t) \qquad \text{if } \gamma \neq \omega, \text{ or}$$

$$x_p = At\cos(\omega t) + Bt\sin(\omega t) \qquad \text{if } \gamma = \omega.$$

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2}\left(\sin(\gamma t) - \frac{\gamma}{\omega}\sin(\omega t)\right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

$$\text{Case (2):} \quad x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t \cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

▸ Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.

Section 12: LRC Series Circuits

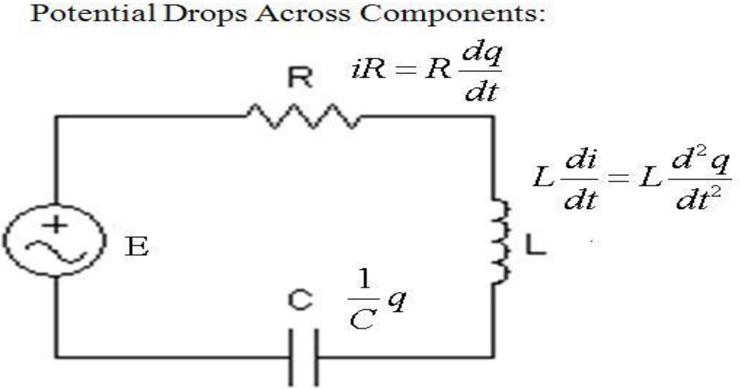

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

This is a second order, linear, constant coefficient nonhomogeneous (if $E \neq 0$) equation.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

$$\begin{array}{lrcl} \textbf{overdamped if} & R^2 - 4L/C & > & 0, \\ \textbf{critically damped if} & R^2 - 4L/C & = & 0, \\ \textbf{underdamped if} & R^2 - 4L/C & < & 0. \end{array}$$

2 real roots

1 real root

complex roots

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system.

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The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system.

Steady state current $i_p = \frac{dq_p}{dt}$

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

The ODE is

$$Lq'' + Rq' + \frac{1}{C}q = E$$

Here, $L = \frac{1}{2}$, $R = 10$, $C = 4 \cdot 10^{-3}$

$$\frac{1}{2}q'' + 10q' + \frac{1}{4 \cdot 10^{-3}}q = 5\cos(10t)$$

We're asked to find steady state current,

$i_p = \frac{dq_p}{dt}$. We'll use undetermined coefficients to find q_p.

$$\frac{1}{4 \cdot 10^{-3}} = \frac{10^3}{4} = \frac{1000}{4} = 250$$

In standard form, the ODE is

$$q'' + 20q' + 500\,q = 10\cos(10t)$$

Find q_c: Characteristic eqn

$$m^2 + 20m + 500 = 0$$

$$m^2 + 20m + 100 - 100 + 500 = 0$$

$$(m+10)^2 = -400$$

$$m + 10 = \pm 20i \Rightarrow m = -10 \pm 20i$$

$$y_c = c_1 e^{-10t}\cos(20t) + c_2 e^{-10t}\sin(20t)$$

Find y_p: $\quad y'' + 20y' + 500y = 10\cos(10t)$

$$g(t) = 10\cos(10t)$$

$$y_p = A\cos(10t) + B\sin(10t)$$

This has no like terms in common w/ y_c
so it is correct.

$$y_p' = -10A\sin(10t) + 10B\cos(10t)$$

$$y_p'' = -100A\cos(10t) - 100B\sin(10t)$$

$$y_p'' + 20y_p' + 500y_p = 10\cos(10t)$$

$$-100A\cos(10t) - 100B\sin(10t) + 20\left(-10A\sin(10t) + 10B\cos(10t)\right)$$

$$+500\left(A\cos(10t) + B\sin(10t)\right)$$

$$= 10\cos(10t) + 0\cdot\sin(10t)$$

zero

Collect $\cos(10t)$ and $\sin(10t)$

$$\cos(10t)\left(-100A + 200B + 500A\right) + \sin(10t)\left(-100B - 200A + 500B\right)$$

$$= 10\cos(10t) + 0\sin(10t)$$

$$400A + 200B = 10$$

$$-200A + 400B = 0 \Rightarrow A = 2B$$

$$40A + 20B = 1 \Rightarrow 40(2B) + 20B = 1$$

$$100B = 1$$

$$B = \frac{1}{100} \quad A = \frac{2}{100}$$

The steady state charge is

$$q_p = \frac{1}{50}\cos(10t) + \frac{1}{100}\sin(10t)$$

The steady state current

$$i_p = \frac{dq_p}{dt} = \frac{-10}{50}\sin(10t) + \frac{10}{100}\cos(10t)$$

$$\boxed{i_p = \frac{1}{10}\cos(10t) - \frac{1}{5}\sin(10t)}$$