October 13 Math 3260 sec. 51 Fall 2025

4.1 Linear Independence

Definition: Linear Independence

The set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in R^m if the homogeneous equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}_m \tag{1}$$

has only the trivial solution, $x_1 = x_2 = \cdots = x_n = 0$, the set is **linearly independent**. If equation (1) has nontrivial solutions, the set is **linearly dependent**.

We saw that

- A set of one vector, $\{\vec{v}\}$ is linearly dependent if and only if $\vec{v} = \vec{0}_m$.
- A set of two vectors $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent if and only if one vector is a scalar multiple of the other.
- For a set of three or more vectors, the equation (1) can be restated in the form $A\vec{x} = \vec{0}_m$.

Example: $\vec{v}_1 = \langle -2, 4, -5 \rangle, \quad \vec{v}_2 = \langle -5, 8, -6 \rangle, \quad \vec{v}_3 = \langle 3, 0, -12 \rangle$

Last time, we set up the matrix A having $\vec{v}_1, \vec{v}_2, \vec{v}_3$ as columns and did row reduction to solve $A\vec{x} = \vec{0}_3$.

$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}_3$

$$[A \,|\, \vec{0}_3] = \left[\begin{array}{rrr|r} -2 & -5 & 3 & 0 \\ 4 & 8 & 0 & 0 \\ -5 & -6 & -12 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrr|r} 1 & 0 & 6 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since A has a non-pivot column, the equation $A\vec{x} = \vec{0}_3$ has nontrivial solutions. The conclusion is that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is **linearly dependent.**

Let's use the results of the rref to give a linear dependence relation.

$\vec{v}_3 = 6\vec{v}_1 + (-3)\vec{v}_2$

$$\vec{v}_1 = \langle -2, 4, -5 \rangle, \quad \vec{v}_2 = \langle -5, 8, -6 \rangle, \quad \vec{v}_3 = \langle 3, 0, -12 \rangle$$

$$6\vec{v}_1 + (-3)\vec{v}_2 = 6\langle -2, 4, -5 \rangle + (-3)\langle -5, 8, -6 \rangle$$

$$= \langle -12, 24, -30 \rangle + \langle 15, -24, 18 \rangle$$

$$= \langle 3, 0, -12 \rangle$$

A linear dependence relation is

$$6\vec{v}_1 - 3\vec{v}_2 - \vec{v}_3 = \vec{0}_3$$

Matrix Columns

Theorem: Let A be an $m \times n$ matrix. The column vectors of A are linearly independent in R^m if and only if the homogeneous equation $A\vec{x} = \vec{0}_m$ has only the trivial solution.

Corollary: Square Matrices & Invertibility

If A is an $n \times n$ matrix, then A is invertible if and only if the column vectors of A are linearly independent.

Remark: Since the invertibility of A implies invertibility of A^T, we can also say that A is invertible if and only if the **row** vectors of A are linearly independent.

Some Linearly Dependent Sets

Theorem: Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a collection of k of vectors in R^n. If

a. one of the vectors, say $\vec{v}_i = \vec{0}_n$, or if

b. $k > n$,

then the collection is linearly dependent.

Remark: Note what this says. It says

- Any set that includes a zero vector is automatically linearly dependent.
- If a set contains more vectors than there are entries in each vector, it's automatically linearly dependent.

Example

Explain why each set below is linearly dependent.

1. $\{\langle 0,0,0\rangle, \langle 1,2,-4\rangle, \langle 5,0,-7\rangle\}$

It contains $\vec{0}_3$.

$1\langle 0,0,0\rangle + 0\langle 1,2,-4\rangle + 0\langle 5,0,-7\rangle = \langle 0,0,0\rangle$

is a lin. dependence relation.

2. $\{\langle 1,-3\rangle, \langle 5,7\rangle, \langle 4,-1\rangle\}$. 3 vectors in $\mathbb{R}^2$, $3>2$.

Let $A = \begin{bmatrix} 1 & 5 & 4 \\ -3 & 7 & -1 \end{bmatrix}$, $A\vec{x} = \vec{0}_2$ has augmented matrix

$\vec{x} = \langle x_1, x_2, x_3\rangle$

$$\left[\begin{array}{ccc|c} 1 & 5 & 4 & 0 \\ -3 & 7 & -1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 1/2 & 0 \end{array}\right]$$

x_3 is free

$x_1 = \frac{-3}{2}x_3$ $\quad$ x_3 - free

$x_2 = \frac{-1}{2}x_3$

so $-\frac{3}{2}x_3\langle 1,-3\rangle + -\frac{1}{2}x_3\langle 5,7\rangle + x_3\langle 4,-1\rangle = \langle 0,0\rangle$ for any choice of x_3. A linear dependence relation is

$$-\frac{3}{2}\langle 1,-3\rangle + \frac{-1}{2}\langle 5,7\rangle + 1\langle 4,-1\rangle = \langle 0,0\rangle$$ (I took $x_3=1$)

Warning

When considering a set of three or more vectors, it's not sufficient to consider them two-at-a-time.

Case in point: $\{\langle 1,-3\rangle, \langle 5,7\rangle, \langle 4,-1\rangle\}$ is **linearly dependent**, but each subset

$$\{\langle 1,-3\rangle, \langle 5,7\rangle\}, \quad \{\langle 1,-3\rangle, \langle 4,-1\rangle\}, \quad \text{and} \quad \{\langle 5,7\rangle, \langle 4,-1\rangle\}$$

is **linearly independent**.

For three or more vectors, linear dependence does mean that at least one vector can be written as a linear combo of the others. But it's not usually obvious just from looking at the vectors.