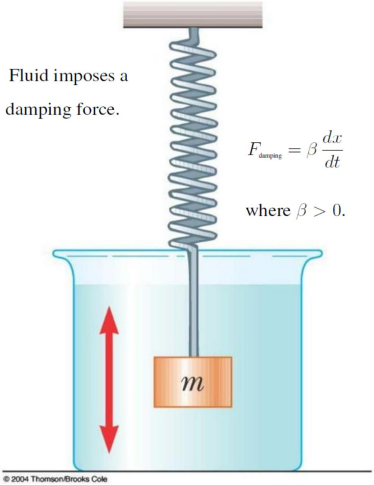
October 14 Math 2306 sec. 52 Fall 2022

Section 11: Linear Mechanical Equations

Free Damped Motion



Free Damped Motion

For object of mass m subjected to spring force with spring constant k and damping force with damping coefficient β, the displacement x satisfies

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

In standard form, the ODE is

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Free Damped Motion

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

This is a second order, linear, constant coefficient, homogeneous ODE with characteristic equation

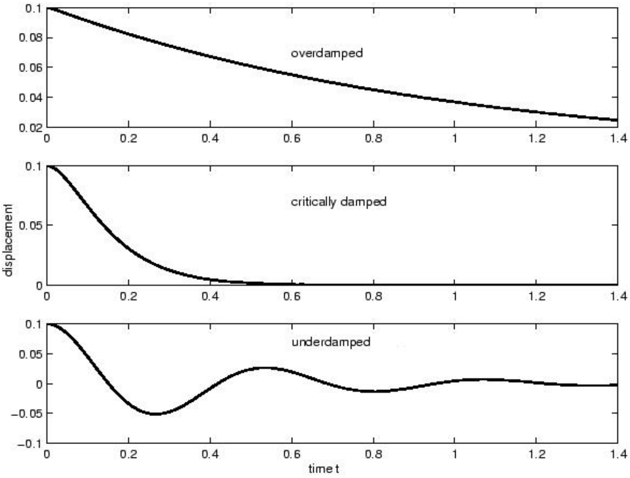
$$r^2 + 2\lambda r + \omega^2 = 0$$

having roots

$$r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation.

Comparison of Damping



2 real

1 real

Complex

Figure: Comparison of motion for the three damping types.

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

$$mx'' + \beta x' + kx = 0$$

$m = 3$ kg

$\beta = 12$

$k = 12$ N/m

$$3x'' + 12x' + 12x = 0$$

Standard form

$$x'' + 4x' + 4x = 0$$

The characteristic equation (in r) is

$$r^2+4r+4=0$$

$$(r+2)^2=0 \implies r=-2 \quad \text{double root}$$

The system is critically damped.

The position

$$x(t)=c_1e^{-2t}+c_2te^{-2t}$$

Apply $x(0)=0 \qquad x'(0)=1$

$$x'(t)=-2c_1e^{-2t}+c_2e^{-2t}-2c_2te^{-2t}$$

$$X(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = 0 \Rightarrow c_1 = 0$$

$$x'(0) = -2c_1 e^0 + c_2 e^0 - 2c_2 \cdot 0 \cdot e^0 = 1$$

$$c_2 = 1$$

$$\boxed{\begin{array}{l}\text{The position is} \\ \qquad x(t) = t e^{-2t}\end{array}}$$

Note: If an object starts "at equilibrium"

then $x(0) = 0$

If it starts "from rest"

then $x'(0) = 0$

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and (2)} \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ Suppose $\gamma \neq \omega$

x_p doesn't have like terms in common w/ x_c,
so this is correct.

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + A\cos(\gamma t) + B_1 \sin(\gamma t)$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ Suppose $\gamma = \omega$

$x_p = A\cos(\omega t) + B\sin(\omega t)$ won't work

$x_p = (A\cos(\omega t) + B\sin(\omega t))t$

$x = c_1\cos(\omega t) + c_2\sin(\omega t) + At\cos(\omega t) + Bt\sin(\omega t)$

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2}\left(\sin(\gamma t) - \frac{\gamma}{\omega}\sin(\omega t)\right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

Case (2): $\quad x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega}t\cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

▸ Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.