October 14 Math 2306 sec. 53 Fall 2024

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2 y'' + xy' - 4y = e^x.$$

Question: Can the method of undetermined coefficients be used to find a particular solution for either of these nonhomogeneous ODEs? (Why/why not?)

The $\tan x$ right side isn't an allowable type,

the second equation is not constant coef.

Variation of Parameters

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x) \tag{1}$$

For the equation (1) in standard form suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

$y_c = c_1 y_1(x) + c_2 y_2(x)$, c_1, c_2 constants

This method is called **variation of parameters**.

u_1 and u_2 are like parameters that vary.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We need a 2^{nd} equation for u_1 and u_2. We'll introduce a 2^{nd} eqn.

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$$

$$\boxed{\text{Assume} \quad u_1' y_1 + u_2' y_2 = 0}$$

Remember that $y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

$$y'' + P(x)y' + Q(x)y = g(x)$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Sub into the ODE

$$u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + P(x)\left(u_1 y_1' + u_2 y_2'\right) + Q(x)\left(u_1 y_1 + u_2 y_2\right) = g(x)$$

Collect u_1', u_2', u_1 and u_2

$$u_1' y_1' + u_2' y_2' + \underbrace{\left(y_1'' + P(x)y_1' + Q(x)y_1\right)}_{=0} u_1 + \underbrace{\left(y_2'' + P(x)y_2' + Q(x)y_2\right)}_{=0} u_2 = g(x)$$

This reduces to

$$u_1' y_1' + u_2' y_2' = g$$ together with

$u_1' y_1 + u_2' y_2 = 0$; we have the system

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g$$

We'll solve this using Cramer's rule. In matrix format, this system is

that is the Wronskian matrix

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

Let $W_1 = \det \begin{bmatrix} 0 & y_2 \\ g & y_2' \end{bmatrix} = 0 - gy_2 = -gy_2$

$W_2 = \det \begin{bmatrix} y_1 & 0 \\ y_1' & g \end{bmatrix} = y_1 g - 0 = y_1 g$

Calling the Wronskian of y_1 and y_2 W

$$u_1' = \frac{W_1}{W} = \frac{-gy_2}{W}$$

$$u_2' = \frac{W_2}{W} = \frac{gy_1}{W}$$

Integrate to get u_1 and u_2, then

$$y_p = u_1 y_1 + u_2 y_2$$

Variation of Parameters

$$y'' + P(x)y' + Q(x)y = g(x)$$

If $\{y_1, y_2\}$ is a fundamental solution set for the associated homogeneous equation, then the general solution is

$$y = y_c + y_p \quad \text{where}$$

$$y_c = c_1 y_1(x) + c_2 y_2(x), \quad \text{and} \quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Letting W denote the Wronskian of y_1 and y_2, the functions u_1 and u_2 are given by the formulas

$$u_1 = \int \frac{-y_2 g}{W}\,dx, \quad \text{and} \quad u_2 = \int \frac{y_1 g}{W}\,dx.$$

Solve the IVP

$$x^2y'' + xy' - 4y = 8x^2, \quad y(1) = 0, \quad y'(1) = 0$$

The complementary solution of the ODE is $y_c = c_1x^2 + c_2x^{-2}$.

We need y_p, we'll use variation of parameters,

$y_p = u_1y_1 + u_2y_2$ where

$$u_1 = \int \frac{-y_2 g}{W}\,dx, \quad \text{and} \quad u_2 = \int \frac{y_1 g}{W}\,dx.$$

From y_c, $y_1 = x^2$ and $y_2 = x^{-2}$

In standard form, $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 8$,

$$g(x) = 8, \quad W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = x^2(-2x^{-3}) - 2x(x^{-2})$$

$$= -4x^{-1}$$

$$u_1 = \int \frac{-g y_2}{w}\, dx = \int \frac{-8x^{-2}}{-4x^{-1}}\, dx = 2\int x^{-1}\, dx = 2\ln x$$

$$u_2 = \int \frac{g y_1}{w}\, dx = \int \frac{8x^2}{-4x^{-1}}\, dx = -2\int x^3\, dx = -2\frac{x^4}{4}$$

$$u_1 = 2\ln x, \quad u_2 = -\frac{1}{2}x^4, \quad y_1 = x^2, \quad y_2 = x^{-2}$$

$$y_p = u_1 y_1 + u_2 y_2 = (2\ln x)x^2 - \frac{1}{2}x^4(x^{-2})$$

$y_p = 2x^2 \ln x - \frac{1}{2}x^2$

$y_c = c_1 x^2 + c_2 x^{-2}$, so the general solution

$$y = c_1 x^2 + c_2 x^{-2} + 2x^2 \ln x - \frac{1}{2}x^2$$

If we let $k_1 = c_1 - \frac{1}{2}$ and $k_2 = c_2$, we can write

$$y = k_1 x^2 + k_2 x^{-2} + 2x^2 \ln x$$

Apply $y(1) = 0$ and $y'(1) = 0$.

$$y' = 2k_1 x - 2k_2 x^{-3} + 4x \ln x + \frac{2x^2}{x}$$

$$y(1) = k_1(1^2) + k_2(1^{-2}) + 2(1^2)\ln 1 = 0$$

$$k_1 + k_2 = 0$$

$$y'(1) = 2k_1(1) - 2k_2(1^{-3}) + 4(1)\ln 1 + 2 = 0$$

$$2k_1 - 2k_2 = -2$$

Solve

$$\begin{aligned} k_1 + k_2 &= 0 \\ 2k_1 - 2k_2 &= -2 \end{aligned} \Rightarrow \begin{aligned} 2k_1 + 2k_2 &= 0 \\ 2k_1 - 2k_2 &= -2 \\ \hline 4k_1 &= -2 \end{aligned}$$

$$k_1 = \frac{-1}{2}$$

$k_2 = -k_1 = \frac{1}{2}$

$$\boxed{\text{The solution to the IVP is} \quad y = \frac{1}{2}x^{-2} - \frac{1}{2}x^2 + 2x^2 \ln x}$$

Two Notes: If we didn't collect like terms before applying the IC, we'd get the same result (we'd find $c_1=0$ and $c_2=1/2$). Also, when doing homework, if your answer is different from the back, but only because you have an added term(s) that is part of y_c, your answer is also correct.