October 15 Math 2306 sec. 54 Fall 2021

Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s,t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_{\alpha}^{\beta} G(s,t)\,dt$$

- the result is a function of the remaining variable s, and
- the variable s is treated as a constant while integrating with respect to t.

For Example...

Assume that $s \neq 0$ and $b > 0$. Compute the integral

$$\int_0^b e^{-st}\,dt$$

Treat s like a constant

$$= \frac{1}{-s} e^{-st} \Big|_0^b$$

$$= \frac{-1}{s} e^{-s(b)} - \frac{-1}{s} e^{-s(0)}$$

$$= \frac{-1}{s} e^{-bs} + \frac{1}{s}$$

$$\int e^{at}\,dt = \frac{1}{a} e^{at} + C$$

a - nonzero constant

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt + \beta\int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The **kernel** for the Laplace transform is $K(s, t) = e^{-st}$.

Limits at Infinity e^{-st}

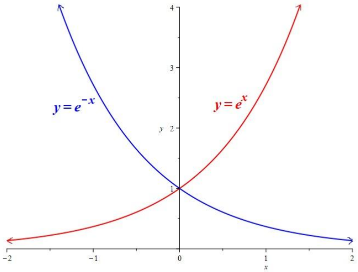
If $s > 0$, evaluate

$$\lim_{t\to\infty} e^{-st}$$

$= 0$

If $s > 0$

$-st \to -\infty$ as $t \to \infty$



If $s < 0$, evaluate

$$\lim_{t\to\infty} e^{-st}$$

$= \infty$

If $s < 0$, then

$-st \to \infty$ as $t \to \infty$

Find the Laplace transform of $f(t) = 1$, $0 \le t < \infty$

By definition

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt$$

We'll consider $s = 0$ and $s \neq 0$.

If $s = 0$ the integral is

$$\int_0^\infty dt = \lim_{b \to \infty} \int_0^b dt = \lim_{b \to \infty} t \Big|_0^b = \lim_{b \to \infty} (b - 0) = \infty$$

The integral is divergent. Zero is not in the domain of $\mathcal{L}\{1\}$. For $s \neq 0$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \, dt = \lim_{b \to \infty} \int_0^b e^{-st} \, dt$$

$$= \lim_{b\to\infty}\left[\frac{-1}{s}e^{-bs} + \frac{1}{s}\right]$$

The integral diverges if

for $s>0$ $\quad = \frac{-1}{s}\cdot 0 + \frac{1}{s}$ $\qquad s \le 0$

$$= \frac{1}{s}$$

Hence $\mathcal{L}\{1\} = \frac{1}{s}$ with domain

$s>0$.

Input $f(t) = 1$ output $F(s) = \frac{1}{s}$

Find the Laplace transform of $f(t) = t$, $0 \le t < \infty$

By definition

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

The integral diverges if $s=0$. For $s \neq 0$,

$$\int_0^\infty e^{-st}\, t\, dt$$

$u = t \qquad du = dt$

$v = \frac{-1}{s} e^{-st} \qquad dv = e^{-st}\, dt$

$$= \frac{-1}{s} t e^{-st} \Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st}\, dt$$

for $s > 0$

For $s > 0$

$$\lim_{t \to \infty} t e^{-st} = 0$$

$$= \frac{-1}{s}(0-0) + \frac{1}{s}\underbrace{\int_0^\infty e^{-st}\,dt}_{\mathcal{L}\{1\}}$$

$$\mathcal{L}\{t\} = \frac{1}{s}\mathcal{L}\{1\}$$

$$= \frac{1}{s}\left(\frac{1}{s}\right)$$

$$= \frac{1}{s^2} \quad \text{for} \quad s>0$$

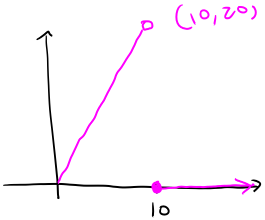
$$\mathcal{L}\{t\} = \frac{1}{s^2} \quad \text{with domain} \quad s>0$$

Input $f(t) = t$ output $F(s) = \frac{1}{s^2}$

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t)=\begin{cases} 2t, & 0\le t<10\\ 0, & t\ge 10\end{cases}$$



$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st} f(t)\,dt + \int_{10}^\infty e^{-st} f(t)\,dt$$

0

$$= \int_0^{10} e^{-st}(2t)\,dt + \int_{10}^\infty e^{-st}(0)\,dt$$

When $s=0$, we have

$$\int_0^{10} 2t\,dt = t^2 \Big|_0^{10} = 100$$

For $s \neq 0$

$$2\int_0^{10} e^{-st}\, t\,dt$$

$$= 2\left[\frac{-1}{s} t e^{-st}\Big|_0^{10} - \frac{1}{s^2} e^{-st}\Big|_0^{10}\right.$$

$$= 2\left(\frac{-10}{s} e^{-10s} - 0 - \frac{1}{s^2}\left(e^{-10s} - e^{0}\right)\right)$$

$$= -\frac{20}{s} e^{-10s} - \frac{2}{s^2} e^{-10s} + \frac{2}{s^2}$$

$$\mathscr{L}\{f(t)\} = \begin{cases} 100, & s=0 \\ \dfrac{2}{s^2} - \dfrac{2}{s^2}e^{-10s} - \dfrac{20}{s}e^{-10s}, & s \neq 0 \end{cases}$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$