October 15 Math 3260 sec. 53 Fall 2025

4.2 Subspaces of R^n

A **subset** of R^n is just some collection of vectors in R^n. We can come up with tons of examples of subsets:

- $B = \{\vec{e}_1, \vec{e}_2\}$ in R^2 is a subset containing the two vectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$.
- The set $W = \{\langle a, b, 0\rangle \mid a, b \in R\}$ is the subset of R^3 of all vectors whose last entry is zero.
- The set $T = \{\langle k, n\rangle \mid k, n \in Z\}$ is the subset of R^2 of all vectors having integer entries.
- The set $P = \mathrm{Span}\{\langle 1, 0, 1, 0\rangle\}$ is the subset of R^4 of all scalar multiples of $\langle 1, 0, 1, 0\rangle$.

4.2 Subspaces of R^n

Not all subsets are created equal. Recall that we have two critical operations in R^n:

- vector addition and
- scalar multiplication.

Subsets of R^n that hold a special sort of importance in Linear Algebra are sets that in some sense preserve these two operations. Such sets are *similar* to R^n in that we can do arithmetic (i.e., use these operations) with vectors in the set and still get vectors in the set.

A set is called **empty** if it doesn't actually contain anything. For example

> *Let M be the set of all nonzero vectors in R^2 that are both parallel to and orthogonal to the vector $\langle 1, 1 \rangle$.*

M is the empty set, "$M = \emptyset$", because no vector satisfies the condition to be in it. Saying a set is **nonempty** just means that there is something in the set.

Definition: Subspace of R^n

A subset, S, of R^n is a **subspace** of R^n provided

i. S is nonempty,
ii. for any pair of vectors $\vec{u}$ and $\vec{v}$ in S, $\vec{u} + \vec{v}$ is in S, and
iii. for any vector $\vec{u}$ in S and scalar c in R, $c\vec{u}$ is in S.

A set that satisfies property

ii. is said to be **closed with respect to vector addition**.
iii. is said to be **closed with respect to scalar multiplication**.

The phrase *"with respect to"* can be repaced with the word *"under"*.

Example: $V = \{\langle a, b\rangle \mid a, b \in R \text{ and } ab \geq 0\}$.

Which of the following vectors is in V?

✓ 1. $\langle 2, 3\rangle$ $a = 2,\ b = 3$ $ab = 6 \geq 0$

✗ 2. $\langle 4, -2\rangle$ $a = 4,\ b = -2$ $ab = -8 < 0$

✓ 3. $\langle -5, -2\rangle$ $a = -5,\ b = -2$ $ab = 10 \geq 0$

✓ 4. $\langle 0, 0\rangle$ $ab = 0 \geq 0$

✓ 5. $\langle -12, 0\rangle$

✗ 6. $\langle -6, 8\rangle$

In the standard, Cartesian coordinate system, a vector in V would have to have standard representation in which quadrant(s)? (I, II, III, or IV)

The coordinate axes are in V. V contains vectors in quadrants I and III.

$V = \{\langle a, b\rangle \mid a, b \in R \text{ and } ab \geq 0\}$.

Suppose $\langle a, b\rangle$ is in V. Is the scalar multiple $c\langle a, b\rangle$ in V is

1. $c > 0$? $c\langle a,b\rangle = \langle ca, cb\rangle$

   $(ca)(cb) = \underset{\overset{\vee}{0}}{c^2}\,\underset{\overset{\vee\vert}{0}}{ab} \geq 0 \Rightarrow c\langle a,b\rangle$ is in V.

2. $c < 0$? $c\langle a,b\rangle = \langle ca, cb\rangle$

   $(ca)(cb) = c^2ab \geq 0 \Rightarrow c\langle a,b)$ is in V.

3. $c = 0$? If $c=0$, $c\langle a,b\rangle = \langle 0,0\rangle$ which is in V.

$V = \{\langle a, b\rangle \mid a, b \in R \text{ and } ab \geq 0\}$.

The following vectors are all in V:

$$\begin{array}{lllllllll} \vec{x}_1 & = & \langle 2,3\rangle, & \vec{x}_3 & = & \langle 0,0\rangle, & \vec{x}_5 & = & \langle -5,-2\rangle, \\ \vec{x}_2 & = & \langle 1,1\rangle, & \vec{x}_4 & = & \langle -1,-1\rangle, & \vec{x}_6 & = & \langle -6,0\rangle, \end{array}$$

Which of the following sums are in V?

$\vec{x}_1 + \vec{x}_2 = \langle 3,4\rangle$ in V

$\vec{x}_3 + \vec{x}_4 = \langle -1,-1\rangle$ in V

$\vec{x}_5 + \vec{x}_6 = \langle -11,-2\rangle$ in V

$\vec{x}_1 + \vec{x}_5 = \langle -3,1\rangle$ not in V

$\vec{x}_2 + \vec{x}_4 = \langle 0,0\rangle$ in V

$\vec{x}_2 + \vec{x}_6 = \langle -5,1\rangle$ not in V

$V = \{\langle a, b\rangle \mid a, b \in R \text{ and } ab \geq 0\}$.

1. Is V a nonempty subset of R^2?

   Yes, $\langle 0,0\rangle$ and $\langle 1,1\rangle$ are vectors in V.

2. Is V closed under scalar multiplication?

   Yes, if $\langle a,b\rangle$ is in V, $c\langle a,b\rangle$ is in V for any scalar c.

3. Is V closed under vector addition?

   No. For example $\langle 2,3\rangle$ and $\langle -5,-2\rangle$ are in V, but $\langle 2,3\rangle + \langle -5,-2\rangle$ is not in V.

4. Is V a subspace of R^2?

   No. V is not closed under vector addition.

$V = \{\langle a, b\rangle \mid a, b \in R \text{ and } ab \geq 0\}$.

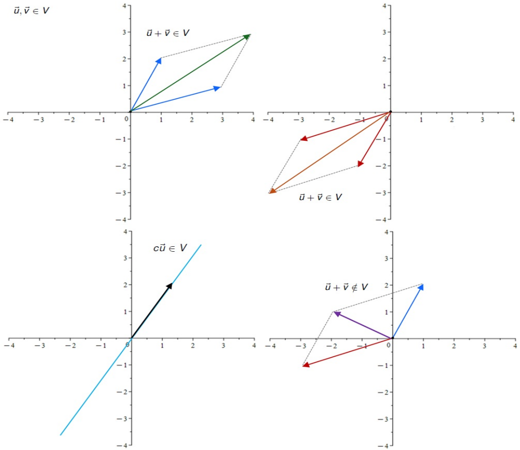

Figure: V is closed under scalar multiplication (lower left), but it is not closed under vector addition (lower right).

Example

Determine whether the following set is a subspace of R^3.

$$W = \{\langle a, b, 0\rangle \mid a, b \in R\}.$$

Is W nonempty—i.e., are there any vectors in R^3 of the form $\langle a, b, 0\rangle$ with a, b some real numbers?

Yes, $\langle 0, 0, 0\rangle$ is in W.

$W = \{\langle a, b, 0\rangle \mid a, b \in R\}$

Is W closed with respect to vector addition?

Let $\vec{u}$ and $\vec{v}$ be in W. Then $\vec{u} = \langle a_1, b_1, 0\rangle$ and $\vec{v} = \langle a_2, b_2, 0\rangle$ for some a_1, a_2, b_1, b_2 in R.

Then

$$\vec{u} + \vec{v} = \langle a_1 + a_2, b_1 + b_2, 0+0\rangle = \langle a_1+a_2, b_1+b_2, 0\rangle$$

$a_1 + a_2$ and $b_1 + b_2$ are real numbers and the third entry of $\vec{u} + \vec{v}$ is zero. So

$\vec{u} + \vec{v}$ is in W.

W is closed under vector addition.

$W = \{\langle a, b, 0\rangle \mid a, b \in R\}$

Is W closed with respect to scalar multiplication?

For $\vec{u} = \langle a_1, b_1, 0\rangle$, let c be any scalar.

$c\vec{u} = \langle ca_1, cb_1, c(0)\rangle = \langle ca_1, cb_1, 0\rangle$.

The third entry is zero, so $c\vec{u}$ is in W.

W is closed under scalar multiplication.

W is a subspace of $\mathbb{R}^3$.

Observation

Note that for any vector $\vec{x} = \langle a, b, 0 \rangle$ in W,

$$\vec{x} = \langle a, b, 0 \rangle = a\langle 1, 0, 0 \rangle + b\langle 0, 1, 0 \rangle.$$

So

$$\vec{x} \in \mathrm{Span}\left\{ \langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle \right\}.$$

Based on how **span** is defined, a set that is defined as a span is closed under both vector addition and scalar multiplication.

Theorem

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is any nonempty subset of vectors in R^n, then the set $\text{Span}(S)$ is a subspace of R^n.

Remark: If a set, say $T = \text{Span}(S)$, is a span, we can call the set of vectors—S in this case—a "*spanning set*."

This gives us two ways to check whether a subset is a subspace or to verify that a subset is a subspace:

1. Use the definition (check that it is nonempty, closed under vector addition, and closed under scalar multiplication). OR

2. Determine whether the set can be written as a span—e.g., look for a spanning set.

Example

Show that the set G defined below is a subspace of R^4 by finding a spanning set.

$$G = \{\langle a+b, a, -b, 3a+2b\rangle \mid a, b \in R\}$$

Let $\vec{x}$ be in G. We want to write $\vec{x}$ as a linear combination of fixed vectors with variable coefficients.

$$\vec{x} = \langle a+b, a, -b, 3a+2b\rangle$$

$$= \underbrace{\langle a, a, 0, 3a\rangle}_{\text{a stuff}} + \underbrace{\langle b, 0, -b, 2b\rangle}_{\text{b stuff}}$$

$$= a\langle 1, 1, 0, 3\rangle + b\langle 1, 0, -1, 2\rangle$$

Since a, b can be any real number,

$$\vec{x} \in \mathrm{Span}\{\langle 1,1,0,3\rangle, \langle 1,0,-1,2\rangle\}.$$

$$G = \mathrm{Span}\{\langle 1,1,0,3\rangle, \langle 1,0,-1,2\rangle\}.$$

So G is a subspace of $\mathbb{R}^4$.

4.2.1 Fundamental Subspaces of a Matrix

We will define four subspaces, two of R^m and two of R^n, associated with an $m \times n$ matrix. Collectively, we call these the **Fundamental Subspaces of a Matrix**.

Column Space

Let A be an $m \times n$ matrix. The subspace of R^m spanned by the column vectors of A, denoted

$$\mathcal{CS}(A) = \mathrm{Span}\{\mathrm{Col}_1(A), \ldots, \mathrm{Col}_n(A)\},$$

is called the **column space of** A.

Remark: We can say

"$\mathcal{CS}(A)$ is the set of all vectors $\vec{y} \in R^m$ such that $A\vec{x} = \vec{y}$ is consistent."

Row Space

Let A be an $m \times n$ matrix. The subspace of R^n spanned by the row vectors of A, denoted

$$\mathcal{RS}(A) = \mathrm{Span}\{\mathrm{Row}_1(A), \ldots, \mathrm{Row}_m(A)\},$$

is called the **row space of** A.

Remark: There is a geometric interpretation of $\mathcal{RS}(A)$, but it will make more sense after we define another fundamental subspace.

Example

Let $A = \begin{bmatrix} 0 & 3 & 1 \\ 4 & 7 & 5 \\ -2 & -5 & -3 \\ 5 & -4 & 2 \end{bmatrix}$. Identify a spanning set for $\mathcal{RS}(A)$ and a spanning set for $\mathcal{CS}(A)$.

The row space is the span of the row vectors, so the set of Row vectors form a spanning set.

$$\{\langle 0,3,1\rangle, \langle 4,7,5\rangle, \langle -2,-5,-3\rangle, \langle 5,-4,2\rangle\}.$$

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 4 & 7 & 5 \\ -2 & -5 & -3 \\ 5 & -4 & 2 \end{bmatrix}$$

A spanning set for CS(A) is the set of column vectors.

$\{\langle 0, 4, -2, 5\rangle, \langle 3, 7, -5, -4\rangle, \langle 1, 5, -3, 2\rangle\}$