October 16 Math 2306 sec. 51 Fall 2024

Section 10: Variation of Parameters

$$y'' + P(x)y' + Q(x)y = g(x)$$

If $\{y_1, y_2\}$ is a fundamental solution set for the associated homogeneous equation, then the general solution is

$$y = y_c + y_p \quad \text{where}$$

$$y_c = c_1 y_1(x) + c_2 y_2(x), \quad \text{and} \quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Letting W denote the Wronskian of y_1 and y_2, the functions u_1 and u_2 are given by the formulas

$$u_1 = \int \frac{-y_2 g}{W}\, dx, \quad \text{and} \quad u_2 = \int \frac{y_1 g}{W}\, dx.$$

Method of Undetermined Coefficients (MUC) -vs- Variation of Parameters (VoP)

Determine which method(s) could be used to find a particular solution for each ODE.

(a) $y'' + 9y = \sec^2(3x)$ V.O.P. only

(b) $y'' + 9y = x^2 \cos(3x)$ V.O.P. or M.U.C.

(c) $y'' - 2y' + y = \dfrac{e^x}{x}$ V.O.P. only

(d) $y'' - 2y' + y = xe^x$ V.O.P. or M.U.C.

Example:

Find the general solution of the ODE $y'' + y = \tan x + 2e^{2x}$.

Let's find y_c first. y_c solves $y'' + y = 0$.

The characteristic equation is

$$r^2 + 1 = 0$$

$$r^2 = -1 \Rightarrow r = \pm\sqrt{-1} = \pm i$$

$$\alpha \pm \beta i \;;\; \alpha = 0, \beta = 1$$

$$y_1 = e^{0x}\cos(1x), \quad y_2 = e^{0x}\sin(1x)$$

$$y_1 = \cos x, \quad y_2 = \sin x$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$y'' + y = \tan x + 2e^{2x}.$$

To find y_p, let's put $y_p = y_{p_1} + y_{p_2}$ where y_{p_1} solves: $y'' + y = \tan x$, and y_{p_2} solves $y'' + y = 2e^{2x}$.

Find y_{p_1} using V.O.P. We have $y_1 = \cos x$, $y_2 = \sin x$. We need g and W.

Here, $g_1(x) = \tan x$.

$$W = \det \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} = \cos x(\cos x) - (-\sin x)\sin x$$

$$= \cos^2 x + \sin^2 x = 1$$

$$y_{p_1} = u_1 y_1 + u_2 y_2$$

$$u_1 = \int -\frac{g y_2}{w}\, dx = \int -\frac{\tan x \sin x}{1}\, dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{(1-\cos^2 x)}{\cos x}\, dx$$

$$= \int (\cos x - \sec x)\, dx$$

$$= \sin x - \ln|\sec x + \tan x|$$

$$\boxed{u_1 = \sin x - \ln|\sec x + \tan x|}$$

$$u_2 = \int \frac{g y_1}{w}\, dx = \int \frac{\tan x \cos x}{1}\, dx = \int \sin x\, dx$$

$$\boxed{u_2 = -\cos x}$$

$y_1 = \cos x$, $y_2 = \sin x$

$$y_{p_1} = u_1 y_1 + u_2 y_2$$

$$= \left(\sin x - \ln|\sec x + \tan x|\right)\cos x + (-\cos x)\sin x$$

$$\boxed{y_{p_1} = -\cos x \ln|\sec x + \tan x|}$$

$y_1 = \cos x$

$y_2 = \sin x$

Find y_{p_2} solving $y'' + y = 2e^{2x}$

using M.U.C.

$g_2(x) = 2e^{2x}$, $y_{p_2} = Ae^{2x}$

which doesn't share any common terms w/ y_c.

Sub in y_{p_2}: $y_{p_2} = Ae^{2x}$, $y_{p_2}' = 2Ae^{2x}$, $y_{p_2}'' = 4Ae^{2x}$

$$y_{p_2}'' + y_{p_2} = 4Ae^{2x} + Ae^{2x} = 2e^{2x}$$

$$5Ae^{2x} = 2e^{2x}$$

$$5A = 2 \Rightarrow A = \frac{2}{5}$$

$$\boxed{y_{p_2} = \frac{2}{5}e^{2x}}$$

$$\boxed{y_{p_1} = -\cos x \ln|\sec x + \tan x|}$$

Hence $y_p = \frac{2}{5}e^{2x} - \cos x \ln|\sec x + \tan x|$

$$\boxed{\text{The general solution to the ODE is} \quad y = C_1 \cos x + C_2 \sin x + \frac{2}{5} e^{2x} - \cos x \ln|\sec x + \tan x|}$$

If we chose V.O.P. for y_{p_2},
we'd have to compute

$$u_1 = \int -2e^{2x} \sin x \, dx \quad , \quad u_2 = \int 2e^{2x} \cos x \, dx$$