October 16 Math 2306 sec. 52 Spring 2023

Section 10: Variation of Parameters

Variation of Parameters

$$y'' + P(x)y' + Q(x)y = g(x)$$

If $\{y_1, y_2\}$ is a fundamental solution set for the associated homogeneous equation, then the general solution is

$$y = y_c + y_p \quad \text{where}$$

$$y_c = c_1 y_1(x) + c_2 y_2(x), \quad \text{and} \quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Letting W denote the Wronskian of y_1 and y_2, the functions u_1 and u_2 are given by the formulas

$$u_1 = \int \frac{-y_2 g}{W}\,dx, \quad \text{and} \quad u_2 = \int \frac{y_1 g}{W}\,dx.$$

Solve the IVP

$$x^2y'' + xy' - 4y = 8x^2, \quad y(1) = 1, \quad y'(1) = 1$$

The complementary solution of the ODE is $y_c = c_1x^2 + c_2x^{-2}$.

We found the particular solution using variation of parameters. With $y_1 = x^2$ and $y_2 = x^{-2}$, the Wronskian turned out to be $W = -4x^{-1}$. Using $g(x) = 8$, we got

$$u_1 = 2\ln(x), \quad \text{and} \quad u_2 = -\frac{1}{2}x^4.$$

Since $y_p = u_1y_1 + u_2y_2$, this gave us

$$y_p = 2x^2\ln(x) - \frac{1}{2}x^2.$$

The general solution $y = y_c + y_p$

$$y = c_1x^2 + c_2x^{-2} + 2x^2\ln x - \frac{1}{2}x^2$$

We can combine $c_1x^2 - \frac{1}{2}x^2 = (c_1 - \frac{1}{2})x^2 = k_1x^2$

Letting $k_2 = c_2$, we can write

$$y = k_1x^2 + k_2x^{-2} + 2x^2\ln x$$

Apply $y(1) = 1$, $y'(1) = 1$

$$y' = 2k_1x - 2k_2x^{-3} + 4x\ln x + 2x^2\left(\frac{1}{x}\right)$$

$$y(1) = k_1(1)^2 + k_2(1)^{-2} + 2(1)^2\ln 1 = 1$$

$$k_1 + k_2 = 1$$

$$y'(1) = 2k_1(1) - 2k_2(1)^{-3} + 4(1)\ln 1 + 2(1) = 1$$

$$2k_1 - 2k_2 + 2 = 1$$

$$2k_1 - 2k_2 = -1$$

Solve $\quad k_1 + k_2 = 1$

$\quad\quad\quad 2k_1 - 2k_2 = -1 \quad \Rightarrow$

$$\begin{aligned} 2k_1 + 2k_2 &= 2 \\ 2k_1 - 2k_2 &= -1 \end{aligned}$$

add $\quad 4k_1 = 1 \Rightarrow k_1 = \frac{1}{4}$

subtract $\quad 4k_2 = 3 \Rightarrow k_2 = \frac{3}{4}$

The solution to the IVP is

$$\boxed{y = \frac{1}{4}x^2 + \frac{3}{4}x^{-2} + 2x^2 \ln x}$$

Method of Undetermined Coefficients (MUC) -vs- Variation of Parameters (VoP)

Determine which method(s) could be used to find a particular solution for each ODE.

(a) $y''+9y=\sec^2(3x)$ VoP is the only option due to the secant on the right side.

(b) $y''+9y=x^2\cos(3x)$ MUC will work, so will VoP

(c) $y''-2y'+y=\dfrac{e^x}{x}$ VoP is the only option due to the x^{-1} factor

(d) $y''-2y'+y=xe^x$ Both methods can be used

Find a Particular Solution

$$y'' - 2y' + y = \frac{e^x}{x}$$

$y_1 = e^x$, $y_2 = xe^x$, $W = e^{2x}$

$u_1 = -x$ $\qquad$ $u_2 = \ln x$

$y_p = u_1 y_1 + u_2 y_2$

$y_p = -xe^x + xe^x \ln x$

Question: Is $y_p = xe^x \ln x$ also correct?

It is since $-xe^{x}$ could be combined

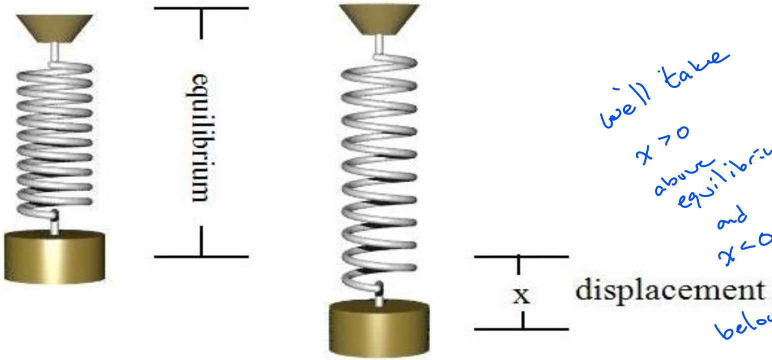
w/ c_2xe^{x} from y_c.

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion**.

▸ Harmonic Motion gif

Building an Equation: Hooke's Law



we'll take $x > 0$ above equilibrium and $x < 0$ below

At equilibrium, displacement $x(t) = 0$.

Hooke's Law: $F_{spring} = k\,x$

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ (mass times acceleration)

$$a = \frac{d^2x}{dt^2} \quad \Longrightarrow \quad F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ (proportional to displacement)

$$m\frac{d^2x}{dt^2} = -kx \quad \Rightarrow \quad m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad \text{let } \omega^2 = \frac{k}{m}$$

x solves $x'' + \omega^2 x = 0$

A 2^{nd} order, linear, homogeneous ODE w/ constant coefficients.

Displacment in Equilibrium

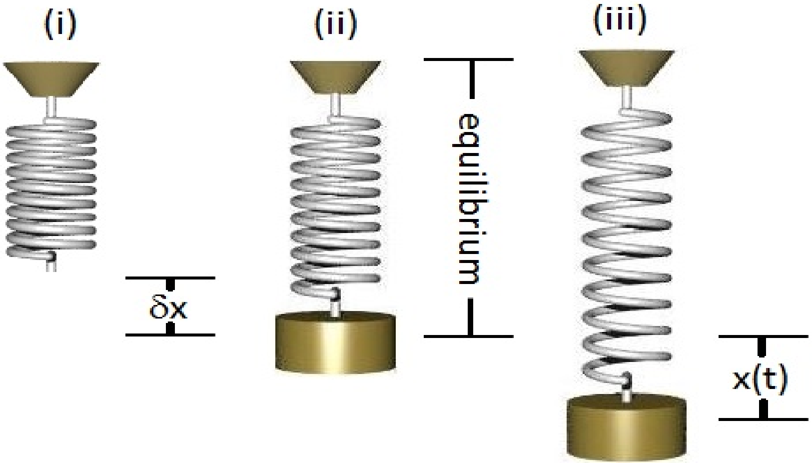


Figure: Spring only, versus spring-mass equilibrium, and spring-mass (nonzero) displacement

Obtaining the Spring Constant (US Customary Units)

If an object with weight W pounds stretches a spring δx feet in equilibrium, then by Hooke's law we compute the spring constant via the equation

$$W = k\delta x.$$

The units for k in this system of measure are lb/ft.

$$k = \frac{W \;\text{lb}}{\delta x \;\text{ft}}$$

Obtaining the Mass (US Customary Units)

Note also that Weight = mass $\times$ acceleration due to gravity. Hence if we know the weight of an object, we can obtain the mass via

$$W = mg.$$

We typically take the approximation $g = 32$ ft/sec^2. The units for mass are lb sec^2/ft which are called slugs.

$m = \dfrac{W}{g}$ slugs

Spring Constant and Mass (SI Units)

In SI units,

- Weight (force) would be in Newtons (N),
- Length would be in meters (m),
- Spring constant would be in N/m
- Mass would be in kilograms (kg)

It is customary to describe an object by its mass in kilograms. When we encounter such a description, we deduce the weight in Newtons

$$W = mg \quad \text{taking the approximation} \quad g = 9.8\,\mathrm{m}/\mathrm{sec}^2.$$