October 17 Math 2306 sec. 51 Fall 2022

Section 12: LRC Series Circuits

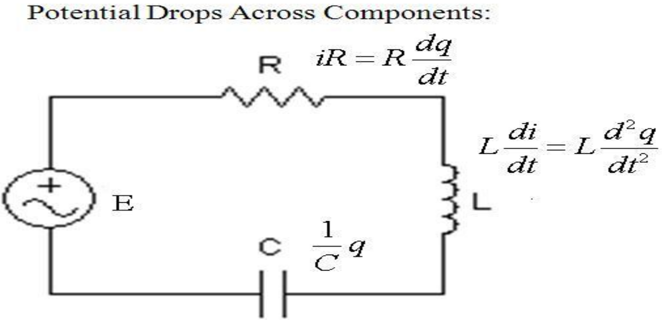

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

This is a second order, linear, constant coefficient nonhomogeneous (if $E \neq 0$) equation.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

overdamped if	$R^2 - 4L/C \ > \ 0,$	(2 real roots)
critically damped if	$R^2 - 4L/C \ = \ 0,$	(1 real root)
underdamped if	$R^2 - 4L/C \ < \ 0.$	(complex roots)

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system.

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Steady State current $i_p = \dfrac{dq_p}{dt}$

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

The model is $Lq'' + Rq' + \frac{1}{C} q = E$

$L = \frac{1}{2}$, $R = 10$, $C = 4 \cdot 10^{-3}$, $E(t) = 5\cos(10t)$

$$\frac{1}{2} q'' + 10 q' + \frac{1}{4 \cdot 10^{-3}} q = 5\cos(10t).$$

We want to find $i_p = \frac{dq_p}{dt}$.

$$\frac{1}{4 \cdot 10^{-3}} = \frac{10^3}{4} = \frac{1000}{4} = 250$$

Standard form:

$$q'' + 20q' + 500q = 10\cos(10t)$$

Let's identify q_c:

$$r^2 + 20r + 500 = 0$$

$$r^2 + 20r + 100 - 100 + 500 = 0$$

$$(r+10)^2 + 400 = 0$$

$$(r+10)^2 = -400$$

$$r+10 = \pm\sqrt{-400} = \pm 20i$$

$$r = -10 \pm 20i$$

$$q_1 = e^{-10t}\cos(20t), \quad q_2 = e^{-10t}\sin(20t)$$

$$q'' + 20q' + 500q = 10\cos(10t)$$

Find q_p using MUC.

$$q_p = A\cos(10t) + B\sin(10t)$$

This is correct.

$500 \quad q_p = A\cos(10t) + B\sin(10t)$

$20 \quad q_p' = -10A\sin(10t) + 10B\cos(10t)$

$1 \quad q_p'' = -100A\cos(10t) - 100B\sin(10t)$

$$q_p'' + 20q_p' + 500q_p = 10\cos(10t)$$

$$\cos(10t)\left(-100A + 200B + 500A\right) + \sin(10t)\left(-100B - 200A + 500B\right)$$

$$= 10\cos(10t) + 0\sin(10t)$$

Matching gives

$$400A + 200B = 10 \qquad\qquad 400A + 200B = 10$$

$$-200A + 400B = 0 \xrightarrow[2\text{ times}]{} -400A + 800B = 0$$

add

$$1000B = 10$$

$$B = \frac{10}{1000} = \frac{1}{100}$$

$$400B = 200A$$

$$A = 2B = 2\left(\frac{1}{100}\right) = \frac{2}{100}$$

The steady state Charge

$$q_p = \frac{2}{100}\cos(10t) + \frac{1}{100}\sin(10t)$$

The steady state current $i_p = \frac{dq_p}{dt}$

$$i_p = \frac{-2}{100}(10)\sin(10t) + \frac{1}{100}(10)\cos(10t)$$

$$\boxed{i_p = \frac{1}{10}\cos(10t) - \frac{2}{10}\sin(10t)}$$