# October 17 Math 2306 sec. 52 Fall 2022

## Section 12: LRC Series Circuits

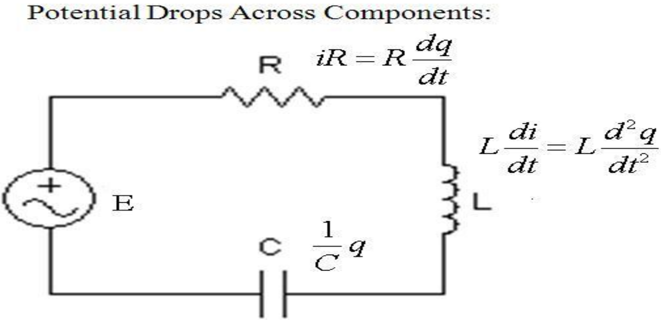


Figure: Kirchhoff's Law: The charge $q$ on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

This is a second order, linear, constant coefficient nonhomogeneous (if $E \neq 0$) equation.

# $LRC$ Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

$$\begin{array}{lrcll} \textbf{overdamped if} & R^2 - 4L/C & > & 0, & \text{(2 real roots)} \\ \textbf{critically damped if} & R^2 - 4L/C & = & 0, & \text{(1 real root)} \\ \textbf{underdamped if} & R^2 - 4L/C & < & 0. & \text{(complex roots)} \end{array}$$

# Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge $q$

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of $q_c$ is influenced by the initial state ($q_0$ and $i_0$) and will decay exponentially as $t \to \infty$. Hence $q_c$ is called the **transient state charge** of the system.

# Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge $q$

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

Steady state current $i_p = \frac{dq_p}{dt}$

The function $q_p$ is independent of the initial state but depends on the characteristics of the circuit ($L$, $R$, and $C$) and the applied voltage $E$. $q_p$ is called the **steady state charge** of the system.

# Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

The ODE is $Lq'' + Rq' + \frac{1}{C}q = E$

$L = \frac{1}{2}$, $R = 10$, $C = 4 \cdot 10^{-3}$, $E = 5\cos(10t)$

$$\frac{1}{2}q'' + 10q' + \frac{1}{4\cdot 10^{-3}}q = 5\cos(10t)$$

we'll find $q_p$ and then $i_p = q_p'$

$$\frac{1}{4\cdot 10^{-3}} = \frac{10^3}{4} = \frac{1000}{4} = 250$$

Standard form

$$q'' + 20q' + 500q = 10\cos(10t)$$

Find $q_c$: $m^2 + 20m + 500 = 0$

$$m^2 + 20m + 100 - 100 + 500 = 0$$

$$(m+10)^2 + 400 = 0$$

$$(m+10)^2 = -400$$

$$m+10 = \pm\sqrt{-400} = \pm 20i$$

$$m = -10 \pm 20i$$

$$q_1 = e^{-10t}\cos(20t), \quad q_2 = e^{-10t}\sin(20t)$$

Find $q_p$ : $\quad q'' + 20q' + 500q = 10\cos(10t)$

$q_p$ Let's use MUC

$500 \qquad q_p = A\cos(10t) + B\sin(10t)$ correct form

$20 \qquad q_p' = -10A\sin(10t) + 10B\cos(10t)$

$1 \qquad q_p'' = -100A\cos(10t) - 100B\sin(10t)$

$$q_p'' + 20q_p' + 500q_p = 10\cos(10t) + 0\sin(10t)$$

$$\cos(10t)\left(-100A + 200B + 500A\right) + \sin(10t)\left(-100B - 200A + 500B\right)$$

$$= 10\cos(10t) + 0\sin(10t)$$

Matching

$$400A + 200B = 10$$
$$-200A + 400B = 0$$

$\xrightarrow[\text{2 times}]{}$

$$400A + 200B = 10$$
$$-400A + 800B = 0$$

add $\quad 1000B = 10$

$$B = \frac{10}{1000} = \frac{1}{100}$$

$$-200A = -400B$$
$$A = 2B = \frac{2}{100}$$

The steady state charge

$$q_p = \frac{2}{100}\cos(10t) + \frac{1}{100}\sin(10t)$$

The steady state current $i_p = q_p'$

$$i_p = \frac{2}{100}(-10)\sin(10t) + \frac{1}{100}(10)\cos(10t)$$

$$\boxed{i_p = \frac{1}{10}\cos(10t) - \frac{2}{10}\sin(10t)}$$