October 17 Math 3260 sec. 51 Fall 2025

4.2.1 Fundamental Subspaces of a Matrix

We defined a **subspace of R^n** as a **nonempty** subset of R^n that is

- closed with respect to vector addition, and
- closed with respect to scalar multiplication.

Theorem

If S is a subspace of R^n, then $\vec{0}_n$ is an element of S.

Remark 1: This does mean that a set that does not contain $\vec{0}_n$ cannot be a subspace of R^n.
Remark 2: Careful! A set can contain $\vec{0}_n$ and still fail to be a subspace. (Consider the example V from class last time.)

Column & Row Spaces of a Matrix

Column Space

Let A be an $m \times n$ matrix. The subspace of R^m spanned by the column vectors of A, denoted

$$\mathcal{CS}(A) = \mathrm{Span}\{\mathrm{Col}_1(A), \ldots, \mathrm{Col}_n(A)\},$$

is called the **column space of** A.

Row Space

Let A be an $m \times n$ matrix. The subspace of R^n spanned by the row vectors of A, denoted

$$\mathcal{RS}(A) = \mathrm{Span}\{\mathrm{Row}_1(A), \ldots, \mathrm{Row}_m(A)\},$$

is called the **row space of** A.

Example

Characterize the column and row spaces of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

$$RS(A) = \text{Span}\{\langle 1,1\rangle, \langle 0,0\rangle\}.$$

$$= \text{Span}\{\langle 1,1\rangle\}$$

$\vec{x}$ is in $RS(A)$ if

$$\vec{x} = c\langle 1,1\rangle = \langle c,c\rangle$$

This would be the vectors whose standard reps live on the 45 degree line.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$CS(A) = \mathrm{Span}\{\langle 1,0\rangle, \langle 1,0\rangle\}$$

$$= \mathrm{Span}\{\langle 1,0\rangle\}$$

$\vec{x}$ is in $CS(A)$ if

$$\vec{x} = a\langle 1,0\rangle = \langle a,0\rangle.$$

This would be all of the vectors
whose standard reps live on the horizontal
axis.

$\mathcal{CS}(A)$ & $\mathcal{RS}(A)$ of $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

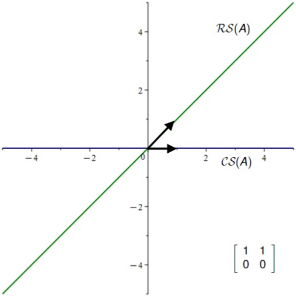

Figure: The row and column spaces of this matrix A are the lines $x_2 = x_1$ and $x_2 = 0$, respectively.

Word of Caution: This is just one example; don't read too much into it. Row and Columns Spaces are not always "lines." In fact, if $A\vec{x} = \vec{y}$ is always consistent, $\mathcal{CS}(A)$ would be all of R^m.