October 17 Math 3260 sec. 53 Fall 2025

4.2.1 Fundamental Subspaces of a Matrix

We defined a **subspace of R^n** as a **nonempty** subset of R^n that is

- closed with respect to vector addition, and
- closed with respect to scalar multiplication.

Theorem

If S is a subspace of R^n, then $\vec{0}_n$ is an element of S.

Remark 1: This does mean that a set that does not contain $\vec{0}_n$ cannot be a subspace of R^n.
Remark 2: Careful! A set can contain $\vec{0}_n$ and still fail to be a subspace. (Consider the example V from class last time.)

Column & Row Spaces of a Matrix

Column Space

Let A be an $m \times n$ matrix. The subspace of R^m spanned by the column vectors of A, denoted

$$\mathcal{CS}(A) = \mathrm{Span}\{\mathrm{Col}_1(A), \ldots, \mathrm{Col}_n(A)\},$$

is called the **column space of** A.

Row Space

Let A be an $m \times n$ matrix. The subspace of R^n spanned by the row vectors of A, denoted

$$\mathcal{RS}(A) = \mathrm{Span}\{\mathrm{Row}_1(A), \ldots, \mathrm{Row}_m(A)\},$$

is called the **row space of** A.

Example

Characterize the column and row spaces of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

$RS(A) = \text{Span}\{\langle 1,1\rangle, \langle 0,0\rangle\}$

$= \text{Span}\{\langle 1,1\rangle\}.$

If $\vec{x}$ is in $RS(A)$, then

$\vec{x} = c\langle 1,1\rangle = \langle c,c\rangle$ for some scalar c.

This is all the 2-tuples with both entries the same number.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$CS(A) = \mathrm{Span}\{\langle 1,0\rangle, \langle 1,0\rangle\}$.

$= \mathrm{Span}\{\langle 1,0\rangle\}$.

If $\vec{y}$ is in $CS(A)$, then

$\vec{y} = a\langle 1,0\rangle = \langle a,0\rangle$.

This is all 2-tuples with second entry zero.

$\mathcal{CS}(A)$ & $\mathcal{RS}(A)$ of $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

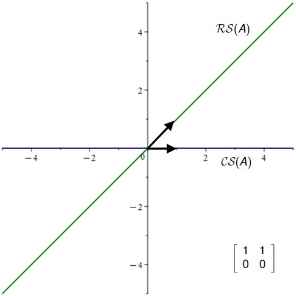

Figure: The row and column spaces of this matrix A are the lines $x_2 = x_1$ and $x_2 = 0$, respectively.

Word of Caution: This is just one example; don't read too much into it. Row and Columns Spaces are not always "lines." In fact, if $A\vec{x} = \vec{y}$ is always consistent, $\mathcal{CS}(A)$ would be all of R^m.

A Third Fundamental Subspace

Definition: Null Space

Let A be an $m \times n$ matrix. The **null space** of A, denoted $\mathcal{N}(A)$, is the set of all solutions of the homogeneous equation $A\vec{x} = \vec{0}_m$. That is,

$$\mathcal{N}(A) = \{\vec{x} \in R^n \mid A\vec{x} = \vec{0}_m\}.$$

- For $m \times n$ matrix A, the product $A\vec{x}$ is only defined if $\vec{x}$ is in R^n.
- The null space contains all solutions of the homogeneous equation $A\vec{x} = \vec{0}_m$.
- To say that $\vec{u} \in \mathcal{N}(A)$ means that $A\vec{u} = \vec{0}_m$.

$\mathcal{N}(A) = \{\vec{x} \in R^n \mid A\vec{x} = \vec{0}_m\}$ is the null space of $m \times n$ matrix A.

Theorem

Let A be an $m \times n$ matrix. Then $\mathcal{N}(A)$ is a subspace of R^n.

Proof: Let A be an $m \times n$ matrix. We have to show that (1) $\mathcal{N}(A)$ is not empty, (2) $\mathcal{N}(A)$ is closed under vector addition, and (3) $\mathcal{N}(A)$ is closed under scalar multiplication.

Since $A\vec{x} = \vec{0}_m$ always admits the trivial solution, $\vec{x} = \vec{0}_n$, $\vec{0}_n$ is in $\mathcal{N}(A)$ making $\mathcal{N}(A)$ nonempty. To show that $\mathcal{N}(A)$ is closed under both operations let $\vec{u}$ and $\vec{v}$ be in $\mathcal{N}(A)$. Then

$A\vec{u} = \vec{0}_m$ and $A\vec{v} = \vec{0}_m$. Note that

$$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0}_m + \vec{0}_m = \vec{0}_m.$$

So $\vec{u}+\vec{v}$ is in $\mathcal{N}(A)$. Let c be any scalar. Note that

$$A(c\vec{u}) = cA\vec{u} = c\vec{0}_m = \vec{0}_m$$

$\mathcal{N}(A)$ is closed under vector addition and scaler multiplication. $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.

Example

Find a spanning set for $\mathcal{N}(A)$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

We need to solve $A\vec{x} = \vec{0}_2$.

set up $[A \mid \vec{0}_2] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

If $\vec{x} = \langle x_1, x_2 \rangle$ then $x_1 = -x_2$.

so $\vec{x} = \langle -x_2, x_2 \rangle = x_2 \langle -1, 1 \rangle$.

$\mathcal{N}(A) = \text{Span}\{\langle -1, 1 \rangle\}$.

Example

Find a spanning set for $\mathcal{N}(A^T)$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $A^T\vec{x} = \vec{0}_2$.

$[A^T \mid \vec{0}_2] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

If $\vec{x} = \langle x_1, x_2 \rangle$ then $x_1 = 0$, x_2 is free

$\vec{x} = \langle 0, x_2 \rangle = x_2 \langle 0, 1 \rangle$

$\mathcal{N}(A^T) = \text{Span}\{\langle 0, 1 \rangle\}$.

Interpreting the Fundamental Subspaces

We already have an interpretation of the column space and the null space. For $m \times n$ matrix A

- $\mathcal{CS}(A)$ is all $\vec{y} \in R^m$ such that $A\vec{x} = \vec{y}$ is consistent, and
- $\mathcal{N}(A)$ is all $\vec{x} \in R^n$ such that $A\vec{x} = \vec{0}_m$.

Question:

How can we interpret the row space?

Since the row space and the null space are both subspaces of R^n, we can ask how they are related. Let's remember that the product

$$A\vec{x} = \langle \text{Row}_1(A) \cdot \vec{x}, \text{Row}_2(A) \cdot \vec{x}, \ldots, \text{Row}_m(A) \cdot \vec{x} \rangle$$

Question from Exam 1

Let $\vec{u}$ and $\vec{v}$ be two vectors in R^n. Suppose $\vec{x}$ is a vector in R^n such that $\vec{x}$ is orthogonal to $\vec{u}$ and $\vec{x}$ is orthogonal to $\vec{v}$. Show that $\vec{x}$ is orthogonal to every vector in $\mathrm{Span}\{\vec{u}, \vec{v}\}$.

This result generalizes. That is, if

$$\vec{x}\cdot\vec{v}_1 = 0, \quad \text{and} \quad \vec{x}\cdot\vec{v}_2 = 0, \quad \text{and} \quad \vec{x}\cdot\vec{v}_3 = 0, \quad \cdots, \quad \text{and} \quad \vec{x}\cdot\vec{v}_m = 0$$

then

$$\vec{x}\cdot\vec{z} = 0$$

for every vector $\vec{z}$ in $\mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_m\}$.

The Row Space

Suppose $\vec{x} \in \mathcal{N}(A)$ for some $m \times n$ matrix A. Then $A\vec{x} = \vec{0}_m$ which means that

$$\begin{aligned}
\text{Row}_1(A) \cdot \vec{x} &= 0\\
\text{Row}_2(A) \cdot \vec{x} &= 0\\
\vdots \quad & \vdots \quad \vdots\\
\text{Row}_m(A) \cdot \vec{x} &= 0
\end{aligned}$$

That is, a vector $\vec{x} \in \mathcal{N}(A)$ is orthogonal to every row vector of A. Since that means that $\vec{x}$ is orthogonal to every linear combination of the row vectors of A, we can say

> Every vector in $\mathcal{RS}(A)$ is orthogonal to every vector in $\mathcal{N}(A)$ and vice versa.

Orthogonal Complements

Let W be a subspace of R^n. The **orthogonal complement** of W, denoted $W^\perp$, is the set of all $\vec{x}$ in R^n that are orthogonal to all vectors in W. We can write

$$W^\perp = \left\{\vec{x} \in R^n \mid \vec{x} \cdot \vec{w} = 0, \ \text{for all} \ \vec{w} \in W\right\}.$$

The symbol $W^\perp$ is read *"W perp."*

$\mathcal{RS}(A)$ & $\mathcal{N}(A)$

For $m \times n$ matrix A, the row space of A is the orthogonal complement of the null space of A.

$$\mathcal{RS}(A) = \mathcal{N}(A)^\perp \quad \text{and} \quad \mathcal{N}(A) = \mathcal{RS}(A)^\perp.$$

$\mathcal{RS}(A)$ & $\mathcal{N}(A)$ of $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

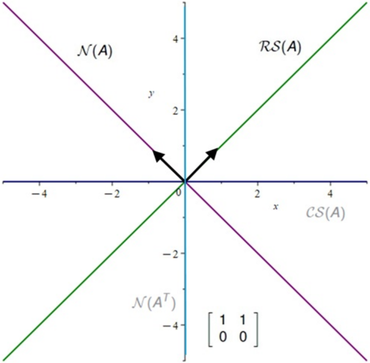

Figure: The row and null spaces of this matrix A are the lines $x_2 = x_1$ and $x_2 = -x_1$, respectively. In this case, they are actually perpendicular lines.

Orthogonal Complements in R^3

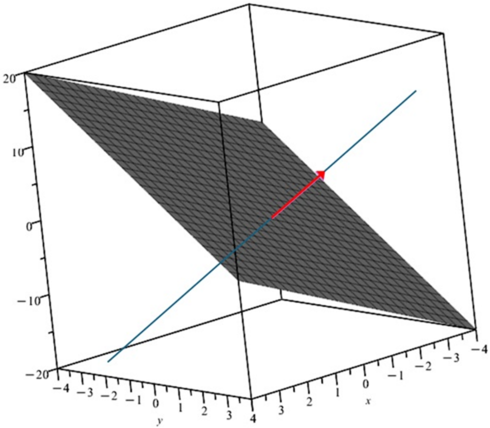

Figure: A subspace of R^3 that corresponds to a plane together with its orthogonal complement corresponding to a line.

The Fourth Fundamental Subspace

The fourth fundamental subspace of a matrix A is the null space of A^T, i.e., $\mathcal{N}(A^T)$. Recall that for a matrix A,

$$\mathrm{Col}_i(A) = \mathrm{Row}_i(A^T) \quad \text{and} \quad \mathrm{Row}_i(A) = \mathrm{Col}_i(A^T).$$

So this fourth subspace is the orthogonal complement of $\mathcal{CS}(A)$.

$\mathcal{N}(A^T)$

For $m \times n$ matrix A

$$\mathcal{N}(A^T) = \left\{ \vec{x} \in R^m \mid A^T\vec{x} = \vec{0}_n \right\}.$$

Equivalently

$$\mathcal{N}(A^T) = \{\vec{x} \in R^m \mid \vec{x} \cdot \vec{y} = 0, \text{ for every } \vec{y} \in \mathcal{CS}(A)\}.$$

$\mathcal{CS}(A)$ & $\mathcal{N}(A^T)$ of $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

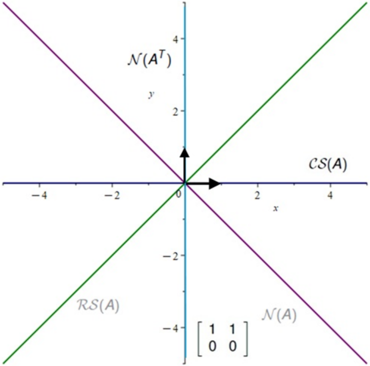

Figure: The column space of A and null space of A^T are the lines $x_2 = 0$ and $x_1 = 0$, respectively. These are also perpendicular line.

Example

Find a spanning set for each of the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 4 \\ 2 & -4 & 1 & -1 \end{bmatrix}.$$

$RS(A) = \text{Span}\{\langle 1,-2,5,4\rangle, \langle 2,-4,1,-1\rangle\}.$

$CS(A) = \text{Span}\{\langle 1,2\rangle, \langle -2,-4\rangle, \langle 5,1\rangle, \langle 4,-1\rangle\}.$

For $\mathcal{N}(A)$ find rref (A).

$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ for $A\vec{x} = \vec{0}_2$

$\vec{x} = \langle x_1, x_2, x_3, x_4\rangle$

$x_1 = 2x_2 + x_4$

$x_3 = -x_4$

x_2, x_4 are free

$\vec{x} = \langle 2x_2 + x_4,\ x_2,\ -x_4,\ x_4 \rangle$

$= x_2 \langle 2, 1, 0, 0 \rangle + x_4 \langle 1, 0, -1, 1 \rangle$

$\mathcal{N}(A) = \mathrm{Span}\{\langle 2,1,0,0\rangle, \langle 1,0,-1,1\rangle\}.$

$\mathcal{N}(A^T)$, we need $\mathrm{rref}(A^T)$

$\mathrm{rref}(A^T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. $\quad A^T\vec{x} = \vec{0}_4$

$\iff \vec{x} = \vec{0}_2$

$\mathcal{N}(A^T) = \mathrm{Span}\{\langle 0,0\rangle\}.$