October 18 Math 2306 sec. 51 Spring 2023

Section 11: Linear Mechanical Equations

We were considering the displacement from equilibrium, $x(t)$, of an object of mass m suspended from a flexible spring with *spring constant* k. In the absence of any sort of damping or external forces, the object exhibits **simple harmonic motion.**

The displacement is subject to the second order, linear, homogeneous differential equation

$$mx'' + kx = 0 \quad \text{i.e.,} \quad x'' + \omega^2 x = 0,$$

where the parameter

$$\omega^2 = \frac{k}{m}.$$

Equilibrium & Displacment in Equilibrium

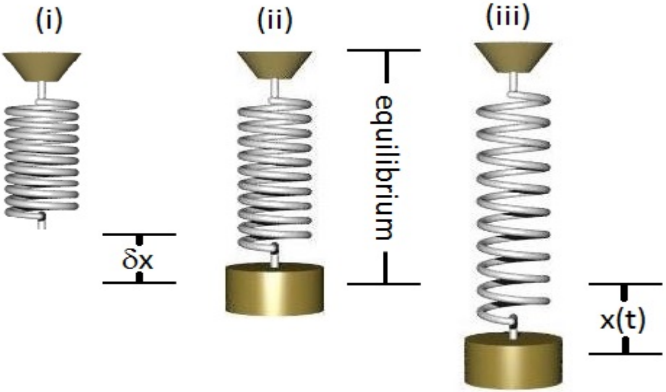


Figure: Hooke's law states that the displacement in equilibrium, δx, is related to the object's weight, W, via $W = k\delta x$. We'll use the convention

$$x > 0 \quad \text{above equilibrium, and} \quad x < 0 \quad \text{below equilibrium.}$$

Basic Units (US Customary)

In US Customary units,

- Force (including weight) would be in pounds (lb),
- Length would be in feet (ft),
- Spring constant would be in lb/ft
- Mass would be in slugs
- Time would be in seconds (sec)

Unless stated otherwise, we'll take the gravitational acceleration constant to be $g = 32$ ft/sec^2. Weight (mg) is generally used as a proxy for mass, so the mass must be computed when needed.

$$m = \frac{W}{g}.$$

Basic Unit (SI Units)

In SI units,

- Force (including weight) would be in Newtons (N),
- Length would be in meters (m),
- Spring constant would be in N/m
- Mass would be in kilograms (kg)
- Time would be in seconds (sec)

It is customary to describe an object by its mass in kilograms. When we encounter such a description, we deduce the weight in Newtons

$$W = mg \quad \text{taking the approximation} \quad g = 9.8\,\mathrm{m}/\mathrm{sec}^2.$$

The *Circular Frequency* ω

Applying Hooke's law with the weight as force, we have

weight

$$mg = k\delta x. \quad \Rightarrow \quad \frac{mg}{m\delta x} = \frac{k\delta x}{m\delta x}$$

We observe that the value ω can be deduced from δx by

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

Provided that values for δx and g are used in appropriate units, ω is in units of per second.

Simple Harmonic Motion

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) \tag{2}$$

called the **equation of motion.**

Caution: The phrase **equation of motion** is used differently by different authors.

Some use this phrase to refer the IVP (1). Others use it to refer to the **solution** to the IVP such as (2).

Simple Harmonic Motion

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$ [1]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[1] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

Amplitude and Phase Shift (alternative definition)

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \cos(\omega t - \hat{\phi})$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and this **phase shift** $\hat{\phi}$ must be defined by

$$\cos \hat{\phi} = \frac{x_0}{A}, \quad \text{with} \quad \sin \hat{\phi} = \frac{x_1}{\omega A}.$$

$\phi + \hat{\phi} = \frac{\pi}{2} + 2\pi n$ for integer n

Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

No damping + no driving $\Rightarrow$ simple harmonic motion

The model should be $mx'' + kx = 0$

$$\Rightarrow \quad x'' + \omega^2 x = 0$$

We need to find ω^2.

$$\omega^2 = \frac{k}{m} \quad \text{and} \quad \omega^2 = \frac{g}{\delta x}$$

We know $\delta x = 6$ inches $= \frac{1}{2}$ ft

Using US units $g = 32 \frac{ft}{sec^2}$

$$\omega^2 = \frac{g}{\delta x} = \frac{32 \frac{ft}{sec^2}}{\frac{1}{2} ft} = 64 \frac{1}{sec^2}$$

The ODE is

$$x'' + 64x = 0$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

We know $\delta x = 6$ in so $\omega^2 = 64 \frac{1}{\text{sec}^2}$, let's find m and k and confirm $\omega^2 = 64$.

The weight $W = 4$ lb, to get mass.

$$W = mg \implies m = \frac{W}{g} = \frac{4\text{ lb}}{32\text{ ft/sec}^2} = \frac{1}{8}\ \frac{\text{lb sec}^2}{\text{ft}}$$

For k,

$$W = k\delta x \implies 4\text{ lb} = k\left(\tfrac{1}{2}\text{ ft}\right) \implies k = 8\ \frac{\text{lb}}{\text{ft}}$$

$$\omega^2 = \frac{k}{m} = \frac{8\ \frac{\text{lb}}{\text{ft}}}{\frac{1}{8}\ \frac{\text{lb sec}^2}{\text{ft}}} = 64\ \frac{1}{\text{sec}^2}$$

The ODE for the displacement is

$$x'' + 64x = 0.$$

Initial conditions

$x(0) = 4$ 4 ft above equilibrium

$x'(0) = -24$ 24 $\frac{\text{ft}}{\text{sec}}$ downward

Let's use the parameter r for the characteristic equation.

$$r^2 + 64 = 0 \implies r^2 = -64 \implies r = \pm 8i$$

$x_1 = \cos(8t)$, $x_2 = \sin(8t)$ so

$$x(t) = c_1 \cos(8t) + c_2 \sin(8t)$$

$$x'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t)$$

$x(0) = c_1 = 4$, $x'(0) = 8c_2 = -24 \implies c_2 = \frac{-24}{8} = -3$

The displacement

$$\boxed{x(t) = 4\cos(8t) - 3\sin(8t)}$$

The period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$ seconds

The linear frequency $f = \frac{1}{T} = \frac{4}{\pi}$ cycles

The amplitude

$$A = \sqrt{4^2 + (-3)^2} = 5 \quad \text{feet}$$

If we write $x(t) = 5\sin(8t + \phi)$, then

ϕ satisfies

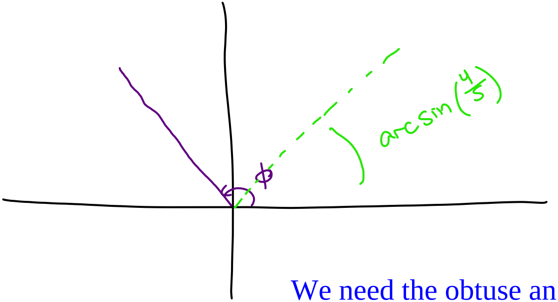
$$\sin\phi = \frac{x_0}{A} = \frac{4}{5} \quad \text{and}$$

$$\cos\phi = \frac{x_1}{\omega A} = \frac{-3}{5}$$

we can find ϕ using the inverse

cosine $\quad \phi = \cos^{-1}\left(\frac{-3}{5}\right)$

$\cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21$ roughly 127°



We need the obtuse angle. If we use the arcsine function, we have to subtract from pi to get the correct phase shift.