October 18 Math 2306 sec. 53 Fall 2024

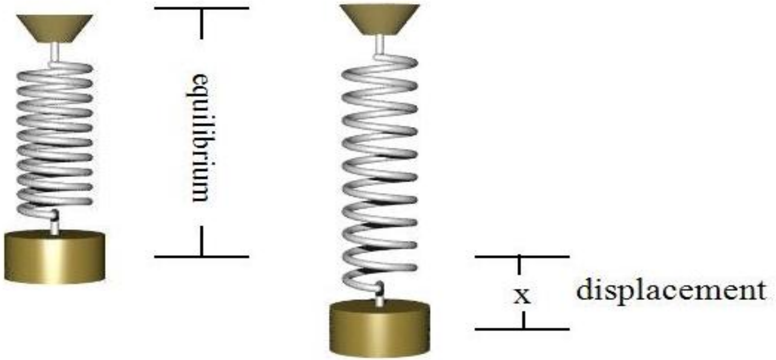
Section 11: Linear Mechanical Equations

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion**.

Harmonic Motion gif

Building an Equation: Hooke's Law



At equilibrium, displacement $x(t) = 0$.

Hooke's Law: $F_{spring} = k\,x$

k is spring constant

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ (mass times acceleration)

$$a = \frac{d^2x}{dt^2} \implies F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ (proportional to displacement)

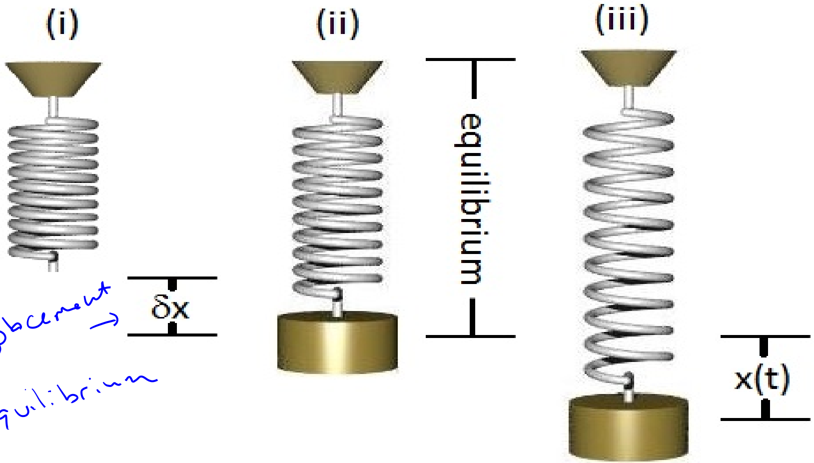
$$m\frac{d^2x}{dt^2} = -kx$$

2nd order, linear, constant coeff, homogeneous ODE.

$mx'' + kx = 0 \Rightarrow x'' + \frac{k}{m}x = 0$, set $\omega^2 = \frac{k}{m}$

So x satisfies $x'' + \omega^2 x = 0$.

Displacment in Equilibrium



displacement in equilibrium

Figure: Spring only, versus spring-mass equilibrium, and spring-mass (nonzero) displacement

Obtaining the Spring Constant (US Customary Units)

We'll use the following basic units when working in US Customary units:

- Force, including weight, in pounds (lb)
- Length in feet (ft)
- Mass in slugs[a] (slug)
- Time in seconds (sec)

[a] Note that one pound $1 \text{ lb} = 1 \, \frac{\text{slug ft}}{\text{sec}^2}$.

If an object with weight W pounds stretches a spring δx feet in equilibrium, then by Hooke's law we compute the spring constant via the equation

$$W = k\delta x \quad \text{i.e.,} \quad k = \frac{W}{\delta x}.$$

The units for k in this system of measure are lb/ft.

Spring Constant and Mass (SI Units)

- Force, including weight, in Newtons (N),
- Length in meters (m),
- Mass in kilograms (kg)
- Time in seconds (sec)

The units for k in this system of measure are N/m.

Weight & Mass

When dealing with US units, weight is usually given in place of mass. In SI, mass is generally stated as mass. We can deduce mass, m, from weight, W, and vice versa via

$$W = mg \quad \text{i.e.,} \quad m = \frac{W}{g},$$

where g is the acceleration due to gravity.

The *Circular Frequency* ω

$$W = mg \quad \text{and} \quad W = k\delta x$$

Applying Hooke's law with the weight as force, we have

$$mg = k\delta x.$$

divide by $m\delta x$

We observe that the value ω can be deduced from δx by

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

Provided that values for δx and g are used in appropriate units, ω is in units of per second.

Simple Harmonic Motion

If x_0 is the initial position of the object (relative to equilibrium) and x_1 is its initial velocity, then the position x satisfies the initial value problem

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Solve this IVP.

The characteristic equation is

$$r^2 + \omega^2 = 0 \Rightarrow r^2 = -\omega^2$$

$$r = \pm\sqrt{-\omega^2} = \pm \omega i \qquad \alpha \pm \beta i \qquad \begin{array}{l} \alpha = 0 \\ \beta = \omega \end{array}$$

$x_1 = \cos(\omega t)$, $x_2 = \sin(\omega t)$

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

$$x(0) = x_0, \quad x'(0) = x_1$$

Apply the I.C. $X' = -\omega c_1 \sin(\omega t) + \omega c_2 \cos(\omega t)$

$X(0) = c_1 \cos 0 + c_2 \sin 0 = x_0 \implies c_1 = x_0$

$X'(0) = -\omega c_1 \sin(0) + \omega c_2 \cos(0) = x_1 \implies c_2 = \dfrac{x_1}{\omega}$

The position @ time t is

$$\boxed{X(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)}$$

Simple Harmonic Motion

The solution to (1),

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t), \tag{2}$$

called the **equation of motion**.

Caution: The phrase **equation of motion** is used differently by different authors.

Some use this phrase to refer the IVP (1). Others use it to refer to the **solution** to the IVP such as (2).

Simple Harmonic Motion

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[1]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

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[1] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

Amplitude and Phase Shift (alternative definition)

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \cos(\omega t - \hat{\phi})$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and this **phase shift** $\hat{\phi}$ must be defined by

$$\cos \hat{\phi} = \frac{x_0}{A}, \quad \text{with} \quad \sin \hat{\phi} = \frac{x_1}{\omega A}.$$

Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

No driving and no damping $\Rightarrow$ simple harmonic motion.

The ODE should look like

$$x'' + \omega^2 x = 0.$$

we need to find ω^2.

We're given displacement in equilibrium

$$\delta x = 6 \text{ in} = \frac{1}{2} \text{ ft}$$

Recall $\omega^2 = \frac{k}{m} = \frac{g}{\delta x}$.

Using $g = 32 \frac{\text{ft}}{\text{sec}^2}$,

$$\omega^2 = \frac{32 \frac{\text{ft}}{\text{sec}^2}}{\frac{1}{2} \text{ ft}} = 64 \frac{1}{\text{sec}^2}$$

The ODE is

$$x'' + 64x = 0$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion in the form $x = A\sin(\omega t + \phi)$, and identify the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

We're given weight $W = 4$ lb and displacement in equilibrium $\delta x = 6$ in $= \frac{1}{2}$ ft.

Let's find k and m.

$$W = k\,\delta x \implies k = \frac{W}{\delta x} = \frac{4\text{ lb}}{\frac{1}{2}\text{ ft}} = 8\ \frac{\text{lb}}{\text{ft}}$$

$$W = mg \implies m = \frac{W}{g} = \frac{4\text{ lb}}{32\ \frac{\text{ft}}{\text{sec}^2}} = \frac{1}{8}\text{ slug}$$

So $\omega^2 = \frac{k}{m} = \frac{8\ \frac{lb}{ft}}{\frac{1}{8}\ \frac{lb}{ft/sec^2}} = 64\ \frac{1}{sec^2}$ as expected

The ODE

$$m x'' + k x = 0 \implies x'' + \omega^2 x = 0$$

$$\frac{1}{8} x'' + 8 x = 0 \implies x'' + 64 x = 0$$

We knew this should be the ODE because the displacement in equilibrium matched the last example.

We'll finish this exercise next time.