October 18 Math 2306 sec. 54 Fall 2021

Section 13: The Laplace Transform

We defined the **Laplace transform.**

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

There is a commonly used lower-case/upper-case convention in which we write

$$\mathscr{L}\{f(t)\} = F(s).$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \dots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

(a) $f(t) = \cos(\pi t)$

$$\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2},$$

Here, $k = \pi$

$$\mathscr{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2}$$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$

$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{2t^4 - e^{-5t} + 3\}$$

$$= 2\mathscr{L}\{t^4\} - \mathscr{L}\{e^{-5t}\} + 3\mathscr{L}\{1\}$$

$$= 2\left(\frac{4!}{s^{4+1}}\right) - \frac{1}{s-(-5)} + 3\left(\frac{1}{s}\right)$$

$$= \frac{2(4!)}{s^5} - \frac{1}{s+5} + \frac{3}{s}$$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

$\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$

$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$

(c) $f(t) = (2-t)^2 = 4 - 4t + t^2$

$\mathscr{L}\{(2-t)^2\} = \mathscr{L}\{4 - 4t + t^2\}$

$= 4\mathscr{L}\{1\} - 4\mathscr{L}\{t\} + \mathscr{L}\{t^2\}$

$= 4\left(\frac{1}{s}\right) - 4\left(\frac{1!}{s^{1+1}}\right) + \frac{2!}{s^{2+1}}$

$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$

* How would I integrate $\int (2-t)^2\,dt$?

Evaluate the Laplace transform $\delta(t-a)$[1]

Suppose δ has the following property: If f is continuous on $[0,\infty)$ and $a \geq 0$, then

$$\int_0^\infty f(t)\delta(t-a)\,dt = f(a).$$

$$\mathscr{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\,\delta(t-a)\,dt$$

← this is of the form with $f(t) = e^{-st}$

$$= e^{-s(a)} = e^{-as}$$

[1]This *function* is called the Dirac delta. It's not really a function in the traditional sense. It's what's known as a *distribution*.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order* c provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

f doesn't $\to \infty$ faster than an exponential.

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0,\infty)$ and of exponential order c for some $c>0$, then f has a Laplace transform for $s>c$.

An example of a function that is NOT of exponential order for any c is $f(t)=e^{t^2}$. Note that

$$f(t)=e^{t^2}=\left(e^t\right)^t \quad \Longrightarrow \quad |f(t)|>e^{ct} \quad \text{whenever} \quad t>c.$$

This is a function that doesn't have a Laplace transform. We won't be dealing with this type of function here.

Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \ldots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Using a Table

When using a table of Laplace transforms, the expression must match exactly. For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$$

so

$$\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = t^3.$$

Note that $n = 3$, so there must be 3! in the numerator and the power $4 = 3 + 1$ on s.

Find the Inverse Laplace Transform

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$$

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^7}\right\}$

If $n+1=7$, then $n=6$.

we need $6!$ on top.

$$\frac{1}{s^7} = \frac{1}{s^7}\cdot\frac{6!}{6!} = \frac{1}{6!}\,\frac{6!}{s^7}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!}\,\frac{6!}{s^7}\right\} = \frac{1}{6!}\,\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{1}{6!}\,t^6$$

Example: Evaluate

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$$

$$\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$$

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{1}{s^2+9}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathscr{L}^{-1}\left\{\frac{3}{3}\frac{1}{s^2+3^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \cos(3t) + \frac{1}{3}\sin(3t)$$

* How would we evaluate

$$\int \frac{s+1}{s^2+9}\,ds$$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

How would we evaluate

$$\int \frac{s-8}{s^2-2s}\,ds\ ?$$

Partial fraction decomp.

$$\frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

Clear fractions

$$s-8 = A(s-2) + Bs$$

$$= (A+B)s - 2A$$

equate like terms

$$A + B = 1 \qquad \Rightarrow B = 1 - A = 1 - 4 = -3$$

$$-2A = -8 \qquad \Rightarrow A = 4$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\frac{s-8}{s(s-2)} = \frac{4}{s} - \frac{3}{s-2}$$

$$\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, \text{ for}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\mathscr{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathscr{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\,\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 3\,\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4 - 3e^{2t}$$