October 19 Math 2306 sec. 52 Fall 2022

Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s,t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_{\alpha}^{\beta} G(s,t)\,dt$$

- the result is a function of the remaining variable s, and
- the variable s is treated as a constant while integrating with respect to t.

For Example...

Assume that $s \neq 0$ and $b > 0$. Compute the integral

$$\int_0^b e^{-st}\,dt = \frac{1}{-s} e^{-st} \Big|_0^b = \frac{1}{-s} e^{-sb} - \frac{1}{-s} e^{0}$$

$$= \frac{1}{s} - \frac{e^{-sb}}{s}$$

treat s like a constant

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt + \beta\int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The **kernel** for the Laplace transform is $K(s, t) = e^{-st}$.

Limits at Infinity e^{-st}

If $s > 0$, evaluate

$$\lim_{t\to\infty} e^{-st} = 0$$

$$-st \to -\infty$$

If $s < 0$, evaluate

$$\lim_{t\to\infty} e^{-st} = \infty$$

$$-st \to +\infty$$

Find the Laplace transform of $f(t) = 1$ $\quad t \geq 0$

By definition

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt$$

If $s=0$, the integral is

$$\int_0^\infty dt = \lim_{b\to\infty} \int_0^b dt = \lim_{b\to\infty} t\Big|_0^b = \lim_{b\to\infty} b = \infty$$

So zero is not in the domain of $\mathcal{L}\{1\}$.

For $s \neq 0$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty} \int_0^b e^{-st}\,dt$$

$$= \lim_{b\to\infty} \frac{1}{s} - \frac{1}{s}e^{-bs}$$

Divergent if $s < 0$.

for $s > 0$

$$\mathcal{L}\{1\} = \lim_{b\to\infty} \frac{1}{s} - \frac{1}{s}e^{-bs} = \frac{1}{s} - 0$$

That is

$$\mathcal{L}\{1\} = \frac{1}{s} \quad \text{for} \quad s > 0$$

Find the Laplace transform of $f(t) = t$ $\quad t \geq 0$

By definition

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

If $s = 0$, the integral is $\int_0^\infty t\, dt$ which diverges.

For $s \neq 0$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

$$= \frac{-1}{s} t e^{-st} \Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st}\, dt$$

Int by parts

$u = t \quad du = dt$

$dv = e^{-st}\, dt$

$v = \frac{-1}{s} e^{-st}$

Convergence requires $s > 0$

$$= 0 - 0 + \frac{1}{s}\underbrace{\int_0^\infty e^{-st}\,dt}_{\mathcal{L}\{1\}} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s > 0$$

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^{10} e^{-st} f(t)\,dt + \int_{10}^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st}(2t)\,dt + \int_{10}^\infty \cancel{e^{-st}(0)}\,dt \quad 0$$

For $s=0$, $\int_0^{10} 2t\,dt = t^2\Big)_0^{10} = 100$

For $s \neq 0$, $$2\int_0^{10} e^{-st}\, t\, dt$$

$u = t \qquad du = dt$

$dv = e^{-st}\, dt$

$v = \frac{-1}{s} e^{-st}$

$$= 2\left(\frac{-1}{s} t e^{-st} \Big|_0^{10} - \int_0^{10} \frac{-1}{s} e^{-st}\, dt \right.$$

$$= 2\left[\frac{-1}{s}(10) e^{-10s} - 0 - \frac{1}{s^2} e^{-st} \Big|_0^{10} \right.$$

$$= 2\left(\frac{-10}{s} e^{-10s} - \frac{1}{s^2} e^{-10s} + \frac{1}{s^2} e^{0} \right)$$

$$= \frac{-20}{s} e^{-10s} - \frac{2}{s^2} e^{-10s} + \frac{2}{s^2}$$

Let $F(s) = \mathcal{L}\{f(t)\}$

$$F(s) = \begin{cases} 100, & s = 0 \\ \frac{2}{s^2} - \frac{20}{s} e^{-10s} - \frac{2}{s^2} e^{-10s}, & s \neq 0 \end{cases}$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

(a) $f(t) = \cos(\pi t)$

$\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}$,

here $k = \pi$

$$\mathscr{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2}$$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

$\mathscr{L}\{1\} = \frac{1}{s}, \quad s$

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}},$

$\mathscr{L}\{e^{at}\} = \frac{1}{s-a},$

$$\mathcal{L}\{f(t)\} = 2\,\mathcal{L}\{t^4\} - \mathcal{L}\{e^{-5t}\} + 3\,\mathcal{L}\{1\}$$

$$= 2\left(\frac{4!}{s^{4+1}}\right) - \frac{1}{s-(-5)} + 3\left(\frac{1}{s}\right)$$

$$= \frac{2(4!)}{s^5} - \frac{1}{s+5} + \frac{3}{s}$$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

$\mathscr{L}\{1\} = \frac{1}{s}$, s

(c) $f(t) = (2-t)^2 = 4 - 4t + t^2$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$,

$$\mathscr{L}\{(2-t)^2\} = 4\mathscr{L}\{1\} - 4\mathscr{L}\{t\} + \mathscr{L}\{t^2\}$$

$$= 4\left(\frac{1}{s}\right) - 4\left(\frac{1}{s^2}\right) + \frac{2!}{s^3}$$

$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$