October 1 Math 2306 sec. 51 Fall 2021

Section 10: Variation of Parameters

We considered a second order, linear, nonhomogeneous equation in standard form

$$y'' + P(x)y' + Q(x)y = g(x),$$

with complementary solution $y_c = c_1y_1 + c_2y_2$. The particular solution found by **variation of parameters** is

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \quad \text{where}$$

$$u_1 = \int \frac{-g(x)y_2(x)}{W(x)}\,dx \quad \text{and} \quad u_2 = \int \frac{g(x)y_1(x)}{W(x)}\,dx.$$

Here, W is the Wronskian of y_1 and y_2.

Example:

Solve the ODE

$$x^2y'' + xy' - 4y = \ln x,$$

given that $y_c = c_1x^2 + c_2x^{-2}$ is the complementary solution.

We need y_p. Using variation of parameters, $y_p = u_1y_1 + u_2y_2$.

We need $g(x)$, $y_1(x)$, $y_2(x)$ and W.

From y_c, $y_1(x) = x^2$ and $y_2(x) = x^{-2}$

Is $g(x) = \ln x$? ODE isn't in standard form, so no.

Standard form: $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2}$

Hence $g_1(x) = \dfrac{\ln x}{x^2}$.

$$W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = x^2(-2x^{-3}) - 2x(x^{-2})$$

$$= -2x^{-1} - 2x^{-1} = -4x^{-1}$$

$$y_1 = x^2, \quad y_2 = x^{-2}, \quad g(x) = \frac{\ln x}{x^2}, \quad W = -4x^{-1}$$

$$u_1 = \int \frac{-g y_2}{W}\,dx = \int \frac{-\frac{\ln x}{x^2}\, x^{-2}}{-4x^{-1}}\,dx = \frac{1}{4}\int (\ln x)\, x^{-2} \cdot x^{-2} \cdot x \, dx$$

$$= \frac{1}{4}\int x^{-3} \ln x \, dx$$

Int by parts

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$dv = x^{-3}\,dx \quad , \quad v = \frac{x^{-2}}{-2}$

$$= \frac{1}{4}\left(\frac{-x^{-2}}{2}\ln x + \int \frac{x^{-2}}{2}\,\frac{1}{x}\,dx\right)$$

$$= \frac{1}{4}\left(\frac{-x^{-2}}{2}\ln x + \frac{1}{2}\int x^{-3}\,dx\right)$$

$$= \frac{1}{4}\left(\frac{-x^{-2}}{2}\ln x - \frac{1}{4}x^{-2}\right)$$

$$\boxed{u_1 = \frac{-x^{-2}}{8}\ln x - \frac{1}{16}x^{-2}}$$

$$u_2 = \int \frac{g y_1}{W}\,dx = \int \frac{\frac{\ln x}{x^2}\cdot x^2}{-4x^{-1}}\,dx = \frac{-1}{4}\int x\ln x\,dx$$

By parts $u = \ln x$, $du = \frac{1}{x}\,dx$

$dv = x\,dx$ $\quad v = \frac{x^2}{2}$

$$= \frac{-1}{4}\left(\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx\right) \qquad \tfrac{1}{2}x$$

$$= \frac{-1}{4}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)$$

$$u_2 = \frac{-1}{8}x^2\ln x + \frac{x^2}{16}$$

$y_1 = x^2$

$y_2 = x^{-2}$

$$\boxed{u_1 = \frac{-x^{-2}}{8}\ln x - \frac{1}{16}x^{-2}}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{-x^{-2}}{8}\ln x - \frac{1}{16}x^{-2}\right)x^2 + \left(\frac{-1}{8}x^2\ln x + \frac{1}{16}x^2\right)x^{-2}$$

$$= \frac{-1}{8}\ln x - \frac{1}{16} - \frac{1}{8}\ln x + \frac{1}{16}$$

$$= \frac{-1}{4}\ln x$$

The general solution is

$$y = c_1 x^2 + c_2 x^{-2} - \frac{1}{4}\ln x$$

Solve the IVP

$$x^2y'' + xy' - 4y = \ln x, \quad y(1) = -1, \quad y'(1) = 0$$

The general solution is

$$y = C_1 x^2 + C_2 x^{-2} - \frac{1}{4}\ln x$$

Apply the initial conditions.

$$y' = 2C_1 x - 2C_2 x^{-3} - \frac{1}{4}\frac{1}{x}$$

$$y(1) = C_1 (1)^2 + C_2 (1)^{-2} - \frac{1}{4}\ln 1 = -1$$

$$C_1 + C_2 = -1$$

$$y'(1) = 2c_1(1) - 2c_2(1)^{-3} - \frac{1}{4} \cdot \frac{1}{1} = 0$$

$$2c_1 - 2c_2 = \frac{1}{4}$$

$$c_1 - c_2 = \frac{1}{8}$$

Solve
$$c_1 + c_2 = -1$$
$$c_1 - c_2 = \frac{1}{8}$$

add
$$2c_1 = \frac{-7}{8} \implies c_1 = \frac{-7}{16}$$

subtract
$$2c_2 = \frac{-9}{8} \implies c_2 = \frac{-9}{16}$$

The solution to the IVP

$$\boxed{y = \frac{-7}{16}x^2 - \frac{9}{16}x^{-2} - \frac{1}{4}\ln x}$$

Section 11: Linear Mechanical Equations

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion**.

Harmonic Motion gif

Building an Equation: Hooke's Law

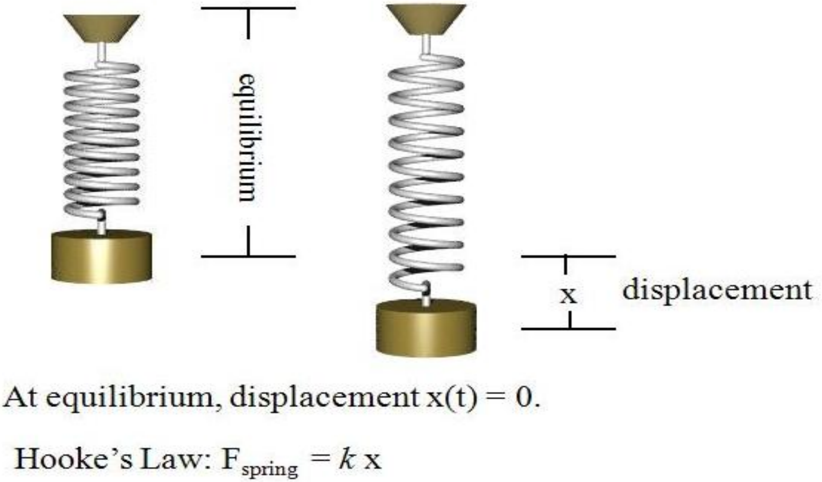

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ (mass times acceleration)

$$a = \frac{d^2x}{dt^2} \implies F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ (proportional to displacement)

$$m\frac{d^2x}{dt^2} = -kx$$

$$\implies mx'' + kx = 0$$

2nd order, linear, homogeneous, constant coefficient ODE.

The ODE is $x'' + \omega^2 x = 0$ where $\omega^2 = \frac{k}{m}$.

Displacment in Equilibrium

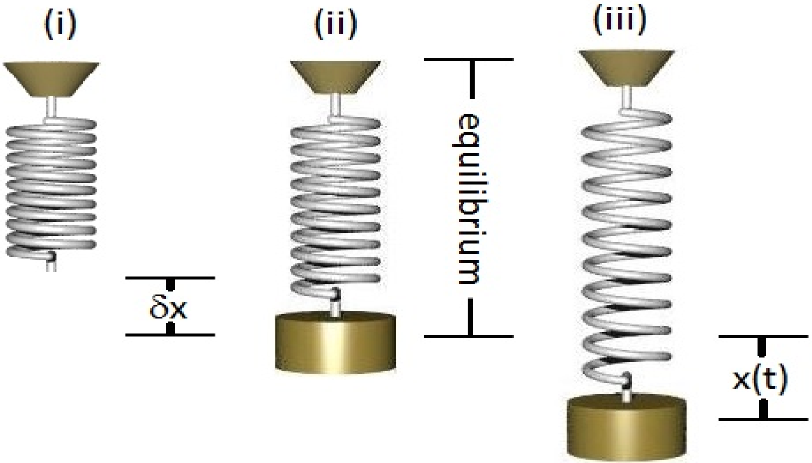

Figure: Spring only, versus spring-mass equilibrium, and spring-mass (nonzero) displacement

Obtaining the Spring Constant (US Customary Units)

If an object with weight $\boxed{W \text{ pounds}}$ stretches a spring $\boxed{\delta x \text{ feet}}$ in equilibrium, then by Hooke's law we compute the spring constant via the equation

$$W = k\delta x.$$

The units for k in this system of measure are lb/ft.

$$k = \frac{W}{\delta x} \qquad \frac{\text{lb}}{\text{ft}}$$

Obtaining the Spring Constant (US Customary Units)

Note also that Weight = mass $\times$ acceleration due to gravity. Hence if we know the weight of an object, we can obtain the mass via

$$W = mg.$$

We typically take the approximation $g = 32$ ft/sec^2. The units for mass are lb sec^2/ft which are called slugs.

$$m = \frac{W}{g}$$

Obtaining the Spring Constant (SI Units)

In SI units,

- Weight (force) would be in Newtons (N),
- Length would be in meters (m),
- Spring constant would be in N/m
- Mass would be in kilograms (kg)

It is customary to describe an object by its mass in kilograms. When we encounter such a description, we deduce the weight in Newtons

$$W = mg \quad \text{taking the approximation} \quad g = 9.8\,\text{m}/\text{sec}^2.$$

The *Circular Frequency* ω

$x'' + \omega^2 x = 0$

Applying Hooke's law with the weight as force, we have

$$mg = k\delta x.$$

We observe that the value ω can be deduced from δx by

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

Provided that values for δx and g are used in appropriate units, ω is in units of per second.

Simple Harmonic Motion

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) \tag{2}$$

called the **equation of motion.**

Caution: The phrase **equation of motion** is used differently by different authors.

Some use this phrase to refer the IVP (1). Others use it to refer to the **solution** to the IVP such as (2).

Simple Harmonic Motion

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[1]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[1] Various authors call f the natural frequency and others use this term for ω.