October 1 Math 2306 sec. 52 Fall 2021

Section 10: Variation of Parameters

We considered a second order, linear, nonhomogeneous equation in standard form

$$y'' + P(x)y' + Q(x)y = g(x),$$

with complementary solution $y_c = c_1y_1 + c_2y_2$. The particular solution found by **variation of parameters** is

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \quad \text{where}$$

$$u_1 = \int \frac{-g(x)y_2(x)}{W(x)}\,dx \quad \text{and} \quad u_2 = \int \frac{g(x)y_1(x)}{W(x)}\,dx.$$

Here, W is the Wronskian of y_1 and y_2.

Example:

Solve the ODE

$$x^2y'' + xy' - 4y = \ln x,$$

given that $y_c = c_1x^2 + c_2x^{-2}$ is the complementary solution.

We need y_p. We'll use variation of parameters.

$y_p = u_1 y_1 + u_2 y_2$. From y_c

$y_1 = x^2$ and $y_2 = x^{-2}$

We need g and W.

Is $g(x) = \ln x$? no, the ODE is not in standard form.

In standard form $\quad y'' + \frac{1}{x} y' - \frac{4}{x^2} y = \frac{\ln x}{x^2}$

Hence $g(x) = \dfrac{\ln x}{x^2}$. $y_1 = x^2$, $y_2 = x^{-2}$

$$W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = x^2(-2x^{-3}) - 2x(x^{-2})$$

$$= -2x^{-1} - 2x^{-1} = -4x^{-1}$$

$$u_1 = \int \frac{-g y_2}{W}\,dx = \int \frac{-\frac{\ln x}{x^2}\, x^{-2}}{-4x^{-1}}\,dx = \frac{1}{4}\int (\ln x)\, x^{-2} \cdot x^{-2} \cdot x\,dx$$

$$= \frac{1}{4}\int x^{-3} \ln x\,dx$$

Int by parts

$u = \ln x$ $\quad du = \frac{1}{x}\,dx$

$dv = x^{-3}\,dx$ $\quad v = \dfrac{-x^{-2}}{2}$

$\frac{1}{2}x^{-3}$

$$= \frac{1}{4}\left(\frac{-x^{-2}}{2}\ln x - \int \frac{-x^{-2}}{2}\cdot\frac{1}{x}\,dx\right).$$

$$= \frac{1}{4}\left(\frac{-x^{-2}}{2}\ln x - \frac{1}{4}x^{-2}\right)$$

$$u_1 = \frac{-x^{-2}}{8}\ln x - \frac{1}{16}x^{-2}$$

$$u_2 = \int \frac{g y_1}{W}\,dx = \int \frac{\frac{\ln x}{x^2}\,x^2}{-4x^{-1}}\,dx = \frac{-1}{4}\int x\ln x\,dx$$

By parts. $u = \ln x \quad du = \frac{1}{x}dx$

$dv = x\,dx \quad v = \frac{x^2}{2}$

$$= \frac{-1}{4}\left(\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx\right) \qquad \frac{1}{2}x$$

$$u_2 = \frac{-1}{4}\left(\frac{x^2}{2}\ln x - \frac{1}{4}x^2\right)$$

$$u_1 = \frac{-x^{-2}}{8}\ln x - \frac{1}{16}x^{-2} \qquad u_2 = \frac{-x^2}{8}\ln x + \frac{1}{16}x^2$$

$$y_1 = x^2 \quad , \quad y_2 = x^{-2}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{-x^{-2}}{8}\ln x - \frac{1}{16}x^{-2}\right)x^2 + \left(\frac{-x^2}{8}\ln x + \frac{1}{16}x^2\right)x^{-2}$$

$$= -\frac{1}{8}\ln x - \frac{1}{16} - \frac{1}{8}\ln x + \frac{1}{16}$$

$$y_p = -\frac{1}{4}\ln x$$

The general solution

$$y = C_1 x^2 + C_2 x^{-2} - \frac{1}{4}\ln x$$

Solve the IVP

$$x^2y'' + xy' - 4y = \ln x, \quad y(1) = -1, \quad y'(1) = 0$$

The general solution to the ODE is

$$y = C_1 x^2 + C_2 x^{-2} - \frac{1}{4}\ln x$$

Apply the initial conditions.

$$y' = 2C_1 x - 2C_2 x^{-3} - \frac{1}{4}\frac{1}{x}$$

$$y(1) = C_1(1)^2 + C_2(1)^{-2} - \frac{1}{4}\ln 1 = -1$$

$$C_1 + C_2 = -1$$

$$y'(1) = 2c_1(1) - 2c_2(1)^{-3} - \frac{1}{4}\cdot\frac{1}{1} = 0$$

$$2c_1 - 2c_2 = \frac{1}{4} \Rightarrow c_1 - c_2 = \frac{1}{8}$$

Solve

$$\begin{aligned} c_1 + c_2 &= -1 \\ c_1 - c_2 &= \frac{1}{8} \end{aligned}$$

add

$$2c_1 = -\frac{7}{8} \Rightarrow c_1 = \frac{-7}{16}$$

subtract

$$2c_2 = \frac{-9}{8} \Rightarrow c_2 = \frac{-9}{16}$$

The solution to the IVP

$$\boxed{y = \frac{-7}{16}x^2 - \frac{9}{16}x^{-2} - \frac{1}{4}\ln x}$$

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion**.

Harmonic Motion gif

Building an Equation: Hooke's Law

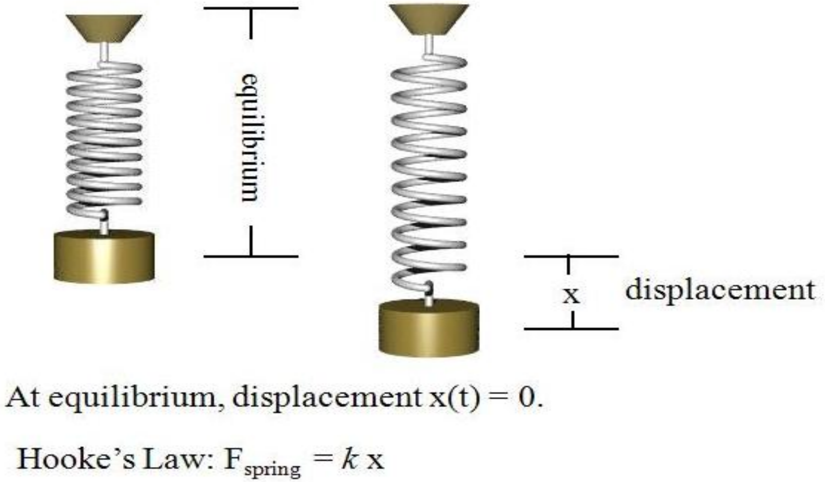

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ (mass times acceleration)

$$a = \frac{d^2x}{dt^2} \implies F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ (proportional to displacement)

$$m\frac{d^2x}{dt^2} = -kx \quad \Rightarrow \quad mx'' + kx = 0$$

2nd order, linear, homogeneous, constant coefficient ODE.

This is $x'' + \omega^2 x = 0$ where $\omega^2 = \frac{k}{m}$

Displacment in Equilibrium

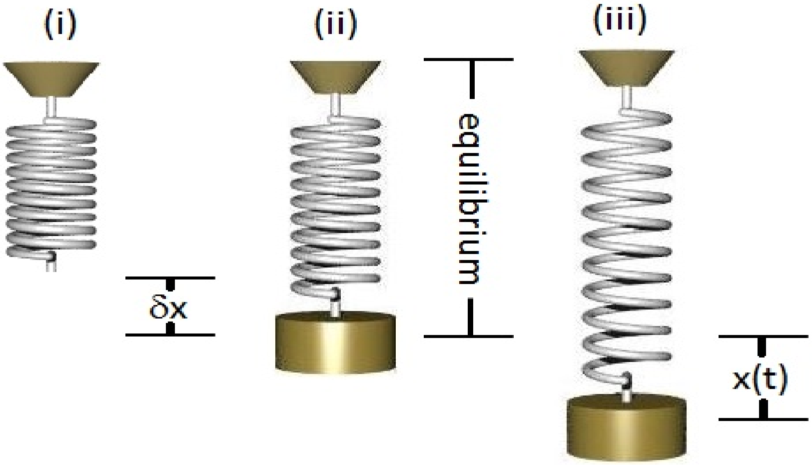

Figure: Spring only, versus spring-mass equilibrium, and spring-mass (nonzero) displacement

Obtaining the Spring Constant (US Customary Units)

If an object with weight W pounds stretches a spring δx feet in equilibrium, then by Hooke's law we compute the spring constant via the equation

$$W = k\delta x.$$

The units for k in this system of measure are lb/ft.

$$k = \frac{W}{\delta x} \sim \frac{\text{lb}}{\text{ft}}$$

Obtaining the Spring Constant (US Customary Units)

Note also that Weight = mass $\times$ acceleration due to gravity. Hence if we know the weight of an object, we can obtain the mass via

$$W = mg.$$

We typically take the approximation $g = 32$ ft/sec^2. The units for mass are lb sec^2/ft which are called slugs.

$$m = \frac{W}{g}$$ slugs

Obtaining the Spring Constant (SI Units)

In SI units,

- Weight (force) would be in Newtons (N),
- Length would be in meters (m),
- Spring constant would be in N/m
- Mass would be in kilograms (kg)

It is customary to describe an object by its mass in kilograms. When we encounter such a description, we deduce the weight in Newtons

$$W = mg \quad \text{taking the approximation} \quad g = 9.8\,\text{m}/\text{sec}^2.$$