October 1 Math 3260 sec. 51 Fall 2025

3.8 Matrix Equations

We'll consider two types of matrix equations.

Matrix-Vector Equation

$$A\vec{x} = \vec{y}$$

The matrix A and the vector $\vec{y}$ are known. The variable to be solved for is the vector $\vec{x}$.

Matrix-Matrix Equation

$$AX = Y$$

The matrices A and Y are known. The variable to be solved for is the matrix X.

Theorem

Suppose that A is an $m \times n$ matrix and that $\vec{y}$ is a vector in R^m and consider the matrix–vector equation

$$A\vec{x} = \vec{y}.$$

Let $\widehat{A}$ be the augmented matrix $\widehat{A} = \left[\begin{array}{c|c} A & \vec{y} \end{array}\right]$.

1. If the rightmost column of $\widehat{A}$ is a pivot column of $\widehat{A}$, then $A\vec{x} = \vec{y}$ is inconsistent.
2. If the rightmost column of $\widehat{A}$ is not a pivot column of $\widehat{A}$, then $A\vec{x} = \vec{y}$ is consistent.

Moreover, if $A\vec{x} = \vec{y}$ is consistent then

1. If every column of A is a pivot column of A, then $A\vec{x} = \vec{y}$ has a unique solution.
2. If at least one column of A is not a pivot column of A, then $A\vec{x} = \vec{y}$ has infinitely many solutions.

Example

Find the solution set of the equation $A\vec{x} = \vec{y}$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -2 & 1 \\ 3 & 4 & -1 \end{bmatrix} \text{ and } \vec{y} = \langle 5, 2, 0 \rangle.$$

$A\vec{x} = \vec{y}$

3×3 $\mathbb{R}^3 \to \mathbb{R}^3$

Using an augmented matrix

$$[A|\vec{y}] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ -1 & -2 & 1 & 2 \\ 3 & 4 & -1 & 0 \end{array}\right] \xrightarrow{\text{rref}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 4 - x_3$

$x_2 = -3 + x_3$

x_3 is free

An rref in Wolfram Alpha

$\vec{x} = \langle x_1, x_2, x_3 \rangle$

Let $x_3 = t$,

$\vec{x} = \langle 4 - t, -3 + t, t \rangle$

$\vec{x} = \langle 4, -3, 0 \rangle + t \langle -1, 1, 1 \rangle, \quad t \in \mathbb{R}$

rref $\left(\left[A \mid \vec{y}\right]\right)$

Consider the equations $A\vec{x} = \vec{y}$, where $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -2 & 1 \\ 3 & 4 & -1 \end{bmatrix}$ and where (a) $\vec{y} = \langle 5, 2, 0\rangle$ or (b) $\vec{y} = \langle 2, 1, 2\rangle$.

$$\text{(a)} \quad \left[A \mid \vec{y}\right] = \left[\begin{array}{rrr|r} 2 & 1 & 1 & 5 \\ -1 & -2 & 1 & 2 \\ 3 & 4 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{(b)} \quad \left[A \mid \vec{y}\right] = \left[\begin{array}{rrr|r} 2 & 1 & 1 & 2 \\ -1 & -2 & 1 & 1 \\ 3 & 4 & -1 & 2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Note that the first three columns of $\text{rref}\left(\left[A \mid \vec{y}\right]\right)$ are the same despite the different augmented column. That part is actually $\text{rref}(A)$.

$\text{rref}\left(\left[A \mid \vec{y}\right]\right)$

If A is an $m \times n$ matrix and $\vec{y}$ is a vector in R^m, then

$$\text{rref}\left(\left[A \mid \vec{y}\right]\right) = \left[\text{rref}(A) \mid \vec{z}\right]$$

where $\vec{z}$ is some vector in R^m that depends on A and $\vec{y}$.

For this example, we have

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since A has a **ROW** with no pivot position, the right most column of $\left[A \mid \vec{y}\right]$ might or might not be a pivot column. It will depend on $\vec{y}$.

Example

Let $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & -1 & 4 & 2 \\ 1 & 0 & 3 & 2 \end{bmatrix}$. Find $\text{rref}(A)$. What can be said about the equation $A\vec{x} = \vec{y}$ for arbitrary vector $\vec{y}$ in R^3?

$A\vec{x} = \vec{y}$ 3×4 R^4

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So for any vector $\vec{y}$

$$\text{rref}\left[A \mid \vec{y}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & z_1 \\ 0 & 1 & -1 & 0 & z_2 \\ 0 & 0 & 0 & 1 & z_3 \end{array}\right]$$

for some z_1, z_2, z_3 in $\mathbb{R}$.

← can't be a pivot column.

$A\vec{x} = \vec{b}$ is consistent for all $\vec{y} \in \mathbb{R}^3$. It will always have ∞-many solutions due to the nonpivot column.

Observation about $\text{rref}\left([A \mid \vec{y}]\right)$

The matrix vector equation

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & -1 & 4 & 2 \\ 1 & 0 & 3 & 2 \end{bmatrix} \vec{x} = \vec{y}$$

will have

$$\text{rref}\left([A \mid \vec{y}]\right) = [\text{rref}(A) \mid \vec{z}] = \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & z_1 \\ 0 & 1 & -1 & 0 & z_2 \\ 0 & 0 & 0 & 1 & z_3 \end{array}\right]$$

Since A has a pivot position in every **ROW**, that right most column can't be a pivot column, no matter what vector $\vec{y}$ we start with.

Theorem

Suppose that A is an $m \times n$ matrix and consider the family of all matrix–vector equations of the form $A\vec{x} = \vec{y}$.

1. If A has a pivot in every row, then $A\vec{x} = \vec{y}$ is consistent for any choice of the vector $\vec{y}$ in R^m.
2. If A does not have a pivot in every row, then there are some vectors $\vec{y}$ in R^m for which $A\vec{x} = \vec{y}$ is consistent and there are also some vectors $\vec{y}$ in R^m for which $A\vec{x} = \vec{y}$ is inconsistent.

Moreover, if $\vec{y}$ is a vector such that $A\vec{x} = \vec{y}$ is consistent then

1. If every column of A is a pivot column of A, then $A\vec{x} = \vec{y}$ has a unique solution.
2. If at least one column of A is not a pivot column of A, then $A\vec{x} = \vec{y}$ has infinitely many solutions.

Remark: Note that this whole theorem is about A, not about any particular augmented matrix.

Key Corollary

If the matrix A happens to be $n \times n$ (i.e., square), this theorem has critical implications for the system $A\vec{x} = \vec{y}$.

Square Matrix Case

Suppose A is an $n \times n$ matrix, and consider the family of equations $A\vec{x} = \vec{y}$.

1. If $\text{rref}(A) = I_n$, then $A\vec{x} = \vec{y}$ has a unique solution for any choice of the vector $\vec{y}$ in R^n.

2. If $\text{rref}\,(A) \neq I_n$, then there are some vectors $\vec{y}$ in R^n for which $A\vec{x} = \vec{y}$ is consistent and there are also some vectors $\vec{y}$ in R^n for which $A\vec{x} = \vec{y}$ is inconsistent. If $A\vec{x} = \vec{y}$ is consistent, then it has infinitely many solutions.

Example

Suppose A and B are 10×10 matrices, $\text{rref}(A) = I_{10}$ and $\text{rref}(B) \neq I_{10}$. What can be said about the homogeneous equations

$$A\vec{x} = \vec{0}_{10} \quad \text{and} \quad B\vec{x} = \vec{0}_{10}$$

Both are homogeneous hence consistent.

Since rref $(A) = I_{10}$, $A\vec{x} = \vec{0}_{10}$ has only the trivial solution.

Since rref $(B) \neq I_{10}$, $B\vec{x} = \vec{0}_{10}$ has non trivial solutions.

Matrix Equation $AX = Y$

Suppose A is a given $m \times n$ matrix and Y is a known $m \times p$ matrix. We can consider the equation

$$AX = Y$$

where X is an $n \times p$ variable matrix.

Note that for $i = 1, \ldots, p$

$$\mathrm{Col}_i(AX) = \mathrm{Col}_i(Y).$$

Since $\mathrm{Col}_i(AX) = A\,\mathrm{Col}_i(X)$, we can consider $AX = Y$ as a **system of matrix-vector equations**

$$\underbrace{A\mathrm{Col}_i(X) = \mathrm{Col}_i(Y)}_{A\vec{x}=\vec{y}}.$$

Solutions $AX = Y$

Let A be $m \times n$, X be $n \times p$ and Y be $m \times p$. We'll say that

- $AX = Y$ is consistent provided each of the p equations $A\,\mathrm{Col}_i(X) = \mathrm{Col}_i(Y)$ is consistent,
- $AX = Y$ is inconsistent if at least one of the equations $A\,\mathrm{Col}_i(X) = \mathrm{Col}_i(Y)$ is inconsistent.

Moreover, if $AX = Y$ is consistent, then $AX = Y$ has a unique solution if and only if each of the p equations $A\,\mathrm{Col}_i(X) = \mathrm{Col}_i(Y)$ has a unique solution

Multiply-Augmented Matrices $AX = Y$

We can study this system of matrix-vector equations using a multiply-augmented matrix. We'll let $\widehat{A}$ be the $m \times (n+p)$ matrix

$$\widehat{A} = [A \mid Y].$$

We'll restrict our attention to the case that A and Y are $n \times n$ matrices. In this case, $\widehat{A}$ will be an $n \times 2n$ matrix. Letting

$$\widehat{A} = [A \mid Y],$$

we find that

$$\text{rref}\left(\widehat{A}\right) = \left[\text{rref}(A) \mid Z\right],$$

where Z is some matrix that depends on A and Y.

Example

Let $A = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$. Set up the 2×4 augmented matrix $[A \mid Y]$ and find its rref.

$$[A \mid Y] = \left[\begin{array}{cc|cc} -1 & -2 & -1 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right] \qquad 2R_2 + R_1 \to R_1$$

$$\begin{array}{cccc} 0 & 2 & -4 & 2 \\ -1 & -2 & -1 & 1 \end{array}$$

$$\left[\begin{array}{cc|cc} -1 & 0 & -5 & 3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$-R_1 \to R_1 \qquad \left[\begin{array}{cc|cc} 1 & 0 & 5 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

Theorem

Suppose that A and Y are both $n \times n$ (square) matrices and consider the matrix equation $AX = Y$. Let

$$\widehat{A} = \left[\begin{array}{c|c} A & Y \end{array}\right]$$

be the $n \times (2n)$ multiply-augmented matrix that corresponds to this matrix equation.

1. If $\operatorname{rref}(A) = I_n$, then $AX = Y$ has a unique solution. Furthermore

   $$\operatorname{rref}\left(\widehat{A}\right) = \left[\begin{array}{c|c} I_n & X \end{array}\right]$$

   where X is the unique solution of $AX = Y$.
2. If $\operatorname{rref}(A) \neq I_n$, then $AX = Y$ is either inconsistent or has infinitely many solutions.

Remark: If any of the n rightmost columns of $\widehat{A}$ is a pivot column, then $AX = Y$ is inconsistent.

Example

Solve the matrix equation $AX = Y$ or show that it is inconsistent.

$$\text{(a)} \quad A = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

Set up $[A|Y] = \left[\begin{array}{cc|cc} -1 & -2 & -1 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{\text{rref}}$

$\left[\begin{array}{cc|cc} 1 & 0 & 5 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$ $AX=Y$ is consistent and

$$X = \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}.$$

Check our work:

$$AX = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = Y$$

Example

Solve the matrix equation $AX = Y$ or show that it is inconsistent.

$$\text{(b)} \quad A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}, \quad \text{and} \quad Y = I_2$$

$$[A|Y] = \left[\begin{array}{cc|cc} 2 & -5 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

$AX = Y$ has unique solution

$$X = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Check:

$$AX = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

✓

Example

Solve the matrix equation $AX = Y$ or show that it is inconsistent.

$$\text{(c)} \quad A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}, \quad \text{and} \quad Y = I_2$$

$$(A|Y) = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ -4 & -6 & 0 & 1 \end{array}\right] \quad 2R_1 + R_2 \to R_2$$

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

$\text{rref}(A) \neq I_2$. Because there is a pivot column on the right side of the bar, $AX = Y$ is inconsistent.

A couple of things to note:

(1) We don't have to check rref(A) before we set up [A|Y]. We'll know whether rref(A) = I_n by doing the row reduction on [A|Y].

(2) If we get a row with all zero on the left side of the bar but with something nonzero on the right side of the bar, we don't have to keep doing row operations. We'll already know that rref(A)≠ I_n and that the equation is not consistent.

If we finished, we'd get $\text{rref}([A \mid I_2]) = \left[\begin{array}{cc|cc} 1 & 3/2 & 0 & -1/4 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$

The extra work doesn't tell us anything we don't already know.