October 20 Math 2306 sec. 51 Spring 2023

Section 11: Linear Mechanical Equations

Free *Undamped* **Motion**

In the absence of damping or external force, the displacement (from equilibrium) of an object of mass m subject to the force of a flexible spring with spring constant k is governed by the second order, linear, homogeneous differential equation

$$mx'' + kx = 0 \quad \text{i.e.,} \quad x'' + \omega^2 x = 0,$$

where the parameter

$$\omega^2 = \frac{k}{m}.$$

Free Damped Motion

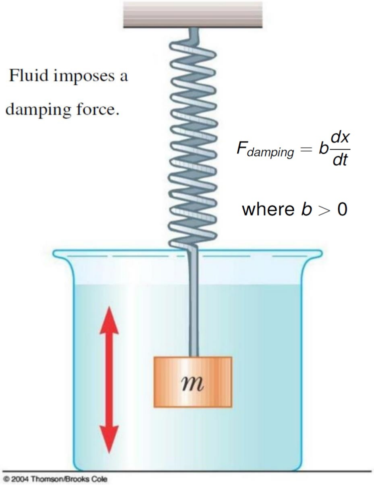

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of spring + Force of damping

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

2^{nd} order, linear, constant coef. homogeneous ODE.

$$mx'' + bx' + kx = 0 \implies x'' + \frac{b}{m}x' + \frac{k}{m}x = 0$$

Let $2\lambda = \frac{b}{m}$ and $\omega^2 = \frac{k}{m}$ the ODE is

$$x'' + 2\lambda x' + \omega^2 x = 0$$

$$x'' + 2\lambda x' + \omega^2 x = 0$$

Using r as the parameter, the characteristic equation is

$$r^2 + 2\lambda r + \omega^2 = 0$$

Completing the square

$$r^2 + 2\lambda r + \lambda^2 = -\omega^2 + \lambda^2$$

$$(r + \lambda)^2 = \lambda^2 - \omega^2$$

$$r = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

There are two real, one real, or complex roots based on the value of $\lambda^2 - \omega^2$.

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

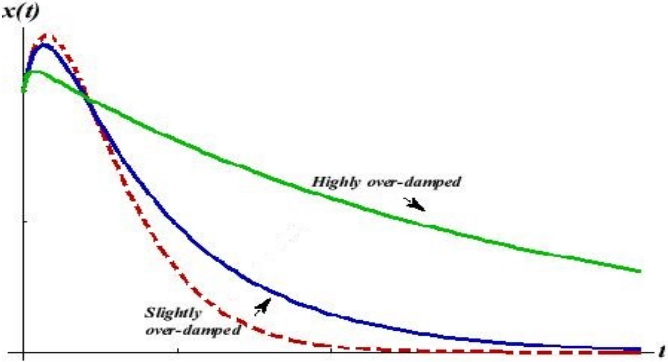

Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

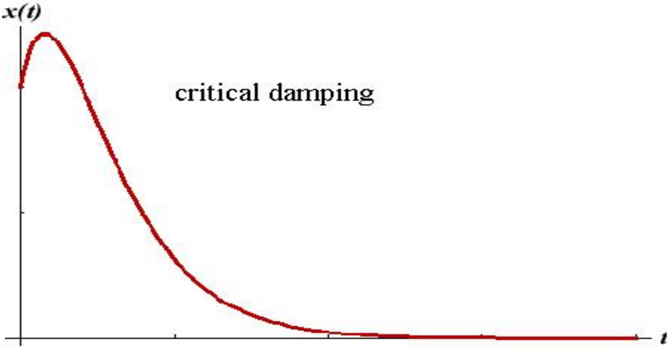

Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

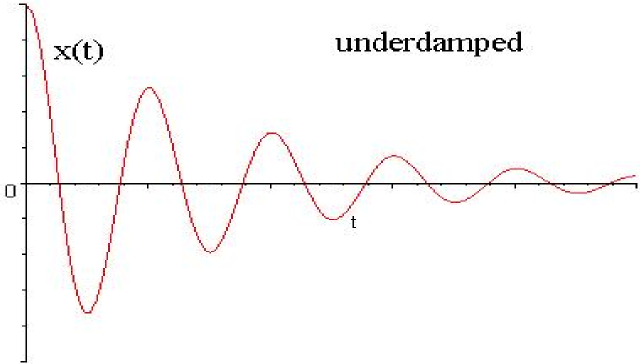

Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping

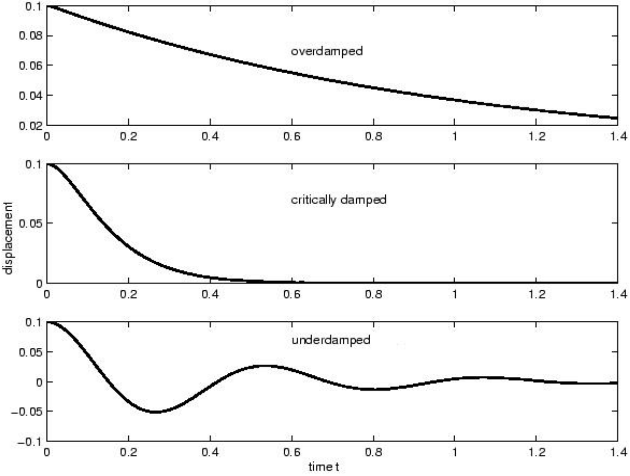

Figure: Comparison of motion for the three damping types.

Example

$* \quad mx'' = -bx' - kx$

$kg \frac{m}{sec^2} \Rightarrow b \sim \frac{kg}{sec}$

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The ODE is $\quad mx'' + bx' + kx = 0$

$m = 2\,kg, \quad k = 12\ N/m, \quad b = 10\ \frac{kg}{sec}$

$$2x'' + 10x' + 12x = 0$$

In standard form $\quad x'' + 5x' + 6x = 0$

w/ characteristic eqn $\quad r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$\Rightarrow$ we get two real roots $r = -2$ or $r = -3$.

$\boxed{\text{The system is over damped.}}$

Note, $\omega^2 = 6$ and $2\lambda = 5 \Rightarrow \lambda = \frac{5}{2}$

$$\lambda^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}, \quad \omega^2 = 6 = \frac{24}{4}$$

$$\lambda^2 - \omega^2 = \frac{1}{4} > 0.$$

Note: If an object is released from equilibrium, then $x(0)=0$

If it starts from rest

then $x'(0)=0$

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

$mx'' + bx' + kx = 0$ Here $m = 3$ kg, $k = 12$ N/m

$b = 12 \frac{kg}{sec}$

The ODE is

$3x'' + 12x' + 12x = 0 \implies x'' + 4x' + 4x = 0$

The characteristic eqn is $r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \implies r = -2$ repeated

$\boxed{\text{The system is critically damped}}$

The general solution to the ODE is

$$x = c_1 e^{-2t} + c_2 t e^{-2t}$$

$x(0)=0$ (released from equilibrium) and

$x'(0) = 1$ ($1 \frac{m}{sec}$ upward velocity).

$$x'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$x(0) = c_1 e^{0} + c_2 \cdot 0 \cdot e^{0} = c_1 = 0$$

$$x'(0) = -2c_1 e^{0} + c_2 e^{0} - 2c_2 \cdot 0 \cdot e^{0} = 1$$

$c_2 = 1$

The displacement

$$x(t) = t\, e^{-2t}$$