October 20 Math 2306 sec. 52 Fall 2021

Section 14: Inverse Laplace Transforms

Recall from section 13 that for f defined on $[0, \infty)$, the Laplace transform of f is defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll use a table to evaluate all inverse Laplace transforms.

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \dots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Evaluate

$\mathscr{L}\{(e^{-t}+e^{2t})^2\}$

Note that

$$(e^{-t}+e^{2t})^2 = (e^{-t})^2 + 2e^{-t}e^{2t} + (e^{2t})^2$$

$$= e^{-2t} + 2e^{t} + e^{4t}$$

$$\mathscr{L}\{(e^{-t}+e^{2t})^2\} = \mathscr{L}\{e^{-2t} + 2e^{t} + e^{4t}\}$$

$$= \mathscr{L}\{e^{-2t}\} + 2\mathscr{L}\{e^{t}\} + \mathscr{L}\{e^{4t}\}$$

$$= \frac{1}{s-(-2)} + 2\,\frac{1}{s-1} + \frac{1}{s-4}$$

$$= \frac{1}{s+2} + \frac{2}{s-1} + \frac{1}{s-4}$$

Evaluate

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-1)(s+4)}\right\}$$

$\mathscr{L}^{-1}\{F(s)G(s)\} \neq \mathscr{L}^{-1}\{F(s)\}\,\mathscr{L}^{-1}\{G(s)\}$

How would you evaluate

$$\int \frac{1}{(s-1)(s+4)}\,ds\ ?$$

Partial fraction decomp.

$$\frac{1}{(s-1)(s+4)} = \frac{A}{s-1} + \frac{B}{s+4}$$

Clear fractions

$$1 = A(s+4) + B(s-1)$$

Set $s = 1$ $\quad 1 = A(5) + B(0) \Rightarrow A = \frac{1}{5}$

$s=-4 \qquad 1 = A(0) + B(-5) \Rightarrow B = \frac{-1}{5}$

$$\frac{1}{(s-1)(s+4)} = \frac{\frac{1}{5}}{s-1} - \frac{\frac{1}{5}}{s+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{5}}{s-1} - \frac{\frac{1}{5}}{s+4}\right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= \frac{1}{5}e^{1t} - \frac{1}{5}e^{-4t}$$

$$= \frac{1}{5}e^{t} - \frac{1}{5}e^{-4t}$$

Find $\mathscr{L}\{f'(t)\}$

Suppose f is of exponential order and has Laplace transform $F(s) = \mathscr{L}\{f(t)\}$. What is

$\mathscr{L}\{f'(t)\}$?

Being of exponential order means that

$$\lim_{t\to\infty} e^{-st} f(t) = 0$$

for s big enough

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty -s e^{-st} f(t)\,dt$$

Int by parts

$dv = f'(t)\,dt$

$v = f(t)$

$u = e^{-st}$

$du = -s e^{-st}\,dt$

$$= 0 - e^{0} f(0) + s \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}\{f'(t)\} = -f(0) + s\,\mathcal{L}\{f(t)\}$$

$$\boxed{\mathcal{L}\{f'(t)\} = s\,F(s) - f(0)}$$

Solve the IVP

$$y' + 4y = e^t, \quad y(0) = 1$$

We'll use the Laplace transform.

We'll assume that $y(t)$ has a Laplace transform, $\mathcal{L}\{y(t)\} = Y(s)$

- take $\mathcal{L}$ of both sides of ODE
- isolate $Y(s)$ using algebra
- find $y(t)$ as $\mathcal{L}^{-1}\{Y(s)\}$.

$$\mathcal{L}\{y' + 4y\} = \mathcal{L}\{e^t\}$$

$$\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{e^t\}$$

$$\mathcal{L}\{y'\} + 4Y(s) = \frac{1}{s-1}$$

$$* \ \mathcal{L}\{y'\} = sY(s) - y(0)$$

$$sY(s) - y(0) + 4Y(s) = \frac{1}{s-1}$$

Use $y(0) = 1$

$$sY(s) - 1 + 4Y(s) = \frac{1}{s-1}$$

Isolate $Y(s)$ using algebra

$$(s+4)Y(s) = \frac{1}{s-1} + 1$$

$$Y(s) = \frac{\frac{1}{s-1}}{s+4} + \frac{1}{s+4}$$

$$Y(s) = \frac{1}{(s-1)(s+4)} + \frac{1}{s+4}$$

Now, we need to take $\mathcal{L}^{-1}\{Y(s)\}$.

From before,

$$\frac{1}{(s-1)(s+4)} = \frac{\frac{1}{5}}{s-1} - \frac{\frac{1}{5}}{s+4}$$

$$Y(s) = \frac{\frac{1}{5}}{s-1} - \frac{\frac{1}{5}}{s+4} + \frac{1}{s+4}$$

$$Y(s) = \frac{\frac{1}{5}}{s-1} + \frac{\frac{4}{5}}{s+4}$$

The solution to the IVP

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= \frac{1}{5}e^{t} + \frac{4}{5}e^{-4t}.$$

$$\boxed{y(t) = \frac{1}{5}e^{t} + \frac{4}{5}e^{-4t}}$$