## October 20 Math 2306 sec. 54 Fall 2021 Section 14: Inverse Laplace Transforms

Recall from section 13 that for *f* defined on  $[0, \infty)$ , the Laplace transform of *f* is defined by

$$\mathscr{L}{f(t)} = \int_0^\infty e^{-st} f(t) \, dt.$$

Now we wish to go *backwards*: Given F(s) can we find a function f(t) such that  $\mathscr{L}{f(t)} = F(s)$ ?

If so, we'll use the following notation

$$\mathscr{L}^{-1}{F(s)} = f(t)$$
 provided  $\mathscr{L}{f(t)} = F(s)$ .

We'll use a table to evaluate all inverse Laplace transforms.

The Laplace Transform is a Linear Transformation

Some basic results include:

$$\blacktriangleright \mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$$

$$\blacktriangleright \mathscr{L}{1} = \frac{1}{s}, \quad s > 0$$

• 
$$\mathscr{L}$$
{ $t^n$ } =  $\frac{n!}{s^{n+1}}$ ,  $s > 0$  for  $n = 1, 2, ...$ 

$$\blacktriangleright \mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

$$\blacktriangleright \mathscr{L}\{\cos kt\} = \frac{s}{s^2 + k^2}, \quad s > 0$$

• 
$$\mathscr{L}{sin kt} = \frac{k}{s^2 + k^2}, \quad s > 0$$

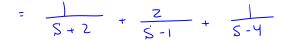
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#### Evaluate

 $\mathscr{L}\{(e^{-t}+e^{2t})^2\}$  $\begin{pmatrix} -t & 2t \\ e & +e \end{pmatrix}^2 = \begin{pmatrix} -t \\ e \end{pmatrix}^2 + 2e e + \begin{pmatrix} e^{2t} \\ e \end{pmatrix}^2$ = -2t + 4t  $\mathcal{L}\left[\left(\bar{e}^{t}+e^{2t}\right)^{2}\right]=\mathcal{L}\left\{\bar{e}^{-2t}+2e^{t}+e^{4t}\right\}$ =  $\mathcal{L}\{e^{2t}\} + 2\mathcal{L}\{e^{t}\} + \mathcal{L}\{e^{4t}\}$ 

 $= \frac{1}{S - (-2)} + 2 \frac{1}{S - 1} + \frac{1}{S - 4}$ October 19, 2021 3/16



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### **Evaluate**

$$\frac{1}{(s-1)(s+4)} = \frac{A}{s-1} + \frac{B}{s+4} \quad Clear fractions$$

$$\int = A(s+4) + B(s-1)$$

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$$= A_{S} + 4A + B_{S} - B$$

$$O_{S} + 1 = (A + B)s + 4A - B$$

$$A + B = 0 \Rightarrow B = -A$$

$$A + B = 0 \Rightarrow B = -A$$

$$YA - (-A) = 1$$

$$SA = 1 \Rightarrow A = \frac{1}{5} - \frac{1}{5}$$

$$\frac{1}{(S - 1)(S + 4)} = \frac{1}{5} - \frac{1}{5} - \frac{1}{5} + \frac{1}{5}$$

$$\frac{1}{(S - 1)(S + 4)} = \frac{1}{5} - \frac{1}{5} - \frac{1}{5} + \frac{1}{5}$$

 $= \frac{1}{5} \mathcal{L}(\frac{1}{5-1}) - \frac{1}{5} \mathcal{L}(\frac{1}{5+4})$ 

 $= \frac{1}{5} e^{t} - \frac{1}{5} e^{-4t}$  $= \pm e^{t} - \pm e^{t}$ 

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## Find $\mathscr{L}{f'(t)}$

 $\mathscr{L}{f'(t)}?$ 

Suppose *f* is of exponential order and has Laplace transform  $F(s) = \mathscr{L}{f(t)}$ . What is

f being of exponential or de

meons lim -st f(t) = 0

$$\mathcal{Z}\left(f'(t)\right) = \int_{0}^{\infty} e^{st} f'(t) dt$$
$$= e^{st} f(t) \int_{0}^{\infty} - \int_{0}^{\infty} e^{st} f(t) dt$$

Int by parts  

$$dv = f'(t) dt$$
  
 $v = f(t)$   
 $u = e^{st}$   
 $du = -se^{st} dt$   
 $e^{st} = se^{st} dt$ 

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$$= O - e^{\circ} f(\circ) + s \int_{0}^{\infty} e^{st} f(t) dt$$

$$Z(f'(t)) = -f(0) + SZ(f(t))$$

$$2(f'(t)) = sF(s) - f(o)$$

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# Solve the IVP Liell use the Laplace $y' + 4y = e^t$ , y(0) = 1transform \* Let L{y(t)]= Y(s) - take & of the ODE - Isolate the Laplace transform $\mathcal{L}\left\{y(t)\right\} = T(s)$ - get ylel by taking L'(Yos). $\chi(y'+4y) = \chi(e^{t})$

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$$\mathcal{L} \{y'\} + 4\mathcal{L} \{y\} = \mathcal{L} \{e^{t}\}$$

$$\mathcal{L} \{y'\} + 4\mathcal{L} \{y\} = \mathcal{L} \{e^{t}\}$$

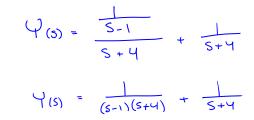
$$\mathcal{L} \{y'\} + 4\mathcal{L} \{y\} = \mathcal{L} \{e^{t}\}$$

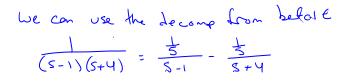
$$\mathcal{L} \{y'\} = \mathcal{L} \{v\}$$

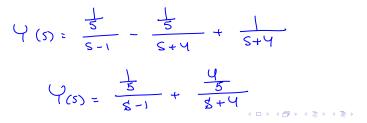
$$\mathcal{L} \{y'\} = \mathcal{L} \{y'\}$$

$$\mathcal{L} \{y'\} = \mathcal{L} \{y'\} = \mathcal{L} \{y'\}$$

$$\mathcal{L} \{y$$







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The solution to the INP  

$$y(t) = \hat{Z}'(Y(s))$$

$$= \frac{1}{5}\hat{Z}'(\frac{1}{5-1}) + \frac{4}{5}\hat{Z}'(\frac{1}{5+4})$$

$$= \frac{1}{5}e^{t} + \frac{4}{5}e^{-4t}$$

$$y(t) = \frac{1}{5}e^{t} + \frac{4}{5}e^{4t}$$

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