October 20 Math 3260 sec. 53 Fall 2025

4.2.1 Fundamental Subspaces of a Matrix

Row & Column Spaces

Let A be an $m \times n$ matrix. The subspace of R^n spanned by the row vectors of A, denoted

$$\mathcal{RS}(A) = \text{Span}\{\text{Row}_1(A), \ldots, \text{Row}_m(A)\},$$

is called the **row space of** A.

The subspace of R^m spanned by the column vectors of A, denoted

$$\mathcal{CS}(A) = \text{Span}\{\text{Col}_1(A), \ldots, \text{Col}_n(A)\},$$

is called the **column space of** A.

These are two of four **fundamental subspaces** of a matrix.

Null Space

Let A be an $m \times n$ matrix. The **null space** of A, denoted $\mathcal{N}(A)$, is the set of all solutions of the homogeneous equation $A\vec{x} = \vec{0}_m$. That is,

$$\mathcal{N}(A) = \{\vec{x} \in R^n \,|\, A\vec{x} = \vec{0}_m\}.$$

The other two fundamental subspaces of a matrix are

- $\mathcal{N}(A)$, which is a subspace of R^n, and
- $\mathcal{N}(A^T)$, which is a subspace of R^m.

$$\text{If } \vec{x} \in \mathcal{N}(A) \quad \text{and} \quad \vec{y} \in \mathcal{RS}(A), \text{ then} \quad \vec{x} \cdot \vec{y} = 0.$$

$$\text{If } \vec{x} \in \mathcal{N}(A^T) \quad \text{and} \quad \vec{y} \in \mathcal{CS}(A), \text{ then} \quad \vec{x} \cdot \vec{y} = 0.$$

Example

Last time, we found that $\mathcal{N}(A) = \text{Span}\{\langle 2,1,0,0\rangle, \langle 1,0,-1,1\rangle\}$, where

$$A = \begin{bmatrix} 1 & -2 & 5 & 4 \\ 2 & -4 & 1 & -1 \end{bmatrix}.$$

Verify that each vector in the spanning set for $\mathcal{N}(A)$ is orthogonal to each row vector of A.

$\mathcal{RS}(A) = \text{Span}\{\langle 1,-2,5,4\rangle, \langle 2,-4,1,-1\rangle\}.$

$\langle 2,1,0,0\rangle \cdot \langle 1,-2,5,4\rangle = 2(1)+1(-2)+0(5)+0(4) = 0$

$\langle 2,1,0,0\rangle \cdot \langle 2,-4,1,-1\rangle = 2(2)+1(-4)+0+0 = 0$

$\langle 1,0,-1,1\rangle \cdot \langle 1,-2,5,4\rangle = 1+(-1)(5)+(4) = 0$

$\langle 1,0,-1,1\rangle \cdot \langle 2,-4,1,-1\rangle = 2+(-1)(1)+(1)(-1) = 0$