October 21 Math 2306 sec. 51 Fall 2024

Section 11: Linear Mechanical Equations

We were considering the displacement from equilibrium, $x(t)$, of an object of mass m suspended from a flexible spring with *spring constant* k. In the absence of any sort of damping or external forces, the object exhibits **simple harmonic motion.**

The displacement is subject to the second order, linear, homogeneous differential equation

$$mx'' + kx = 0 \quad \text{i.e.,} \quad x'' + \omega^2 x = 0,$$

where the parameter

$$\omega^2 = \frac{k}{m}.$$

Equilibrium & Displacment in Equilibrium

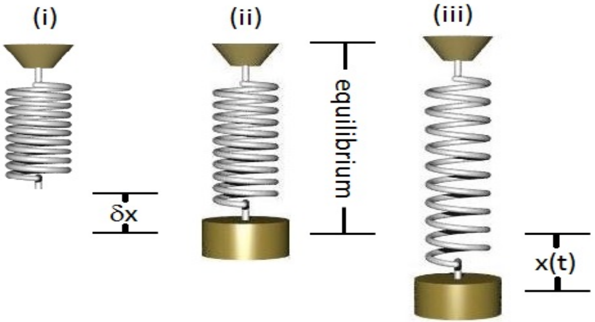


Figure: Hooke's law states that the displacement in equilibrium, δx, is related to the object's weight, W, via $W = k\delta x$. We'll use the convention

$$x > 0 \quad \text{above equilibrium, and} \quad x < 0 \quad \text{below equilibrium.}$$

Remark: Since $W = mg = k\delta x$, $\omega^2 = \dfrac{k}{m} = \dfrac{g}{\delta x}$.

Simple Harmonic Motion

With initial displacement x_0 and initial velocity, x_1, the position $x(t)$ at time t satisfies the IVP

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1.$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[1]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[1] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi) = A \cos(\omega t - \hat{\phi})$$

where the amplitude

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ or $\hat{\phi}$ must be defined by

$$\sin\phi = \frac{x_0}{A}, \text{ and } \cos\phi = \frac{x_1}{\omega A} \quad \text{or} \quad \cos\hat{\phi} = \frac{x_0}{A}, \text{ and } \sin\hat{\phi} = \frac{x_1}{\omega A}.$$

Note, $\phi + \hat{\phi} = \frac{\pi}{2}$ (up to an integer multiple of 2π).

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion in the form $x = A\sin(\omega t + \phi)$, and identify the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

From the weight $W = 4$lb and displacement in equilibrium $\delta x = \frac{1}{2}$ ft, we found the mass and spring constant

$$m = \frac{1}{8}\text{ slugs,} \quad \text{and} \quad k = 8\,\frac{\text{lb}}{\text{ft}}.$$

This makes

$$\omega^2 = 64\,\frac{1}{\text{sec}^2}.$$

We ended up with the ODE

$$\frac{1}{8}x'' + 8x = 0 \quad \Longrightarrow \quad x'' + 64x = 0.$$

We need $x(0)$ and $x'(0)$

$x(0) = 4$ (4 ft above equilibrium)

$x'(0) = -24$ (24 ft/sec downward)

The characteristic equation $r^2 + 64 = 0$

$r^2 = -64 \Rightarrow r = \pm 8i$

$x(t) = c_1 \cos(8t) + c_2 \sin(8t)$

$x'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t)$

$x(0) = c_1 \cos(0) + c_2 \sin(0) = 4 \Rightarrow c_1 = 4$

$x'(0) = -8c_1 \sin(0) + 8c_2 \cos(0) = -24 \Rightarrow 8c_2 = -24$

$c_2 = -3$

$x(t) = 4\cos(8t) - 3\sin(8t)$

Note that amplitude $A = \sqrt{4^2 + (-3)^2} = 5$

So

$x(t) = 5\sin(8t + \phi)$ where

$\sin\phi = \frac{4}{5}$ and $\cos\phi = \frac{-3}{5}$

The period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

Linear frequency $f = \frac{4}{\pi}$

The phase shift $\phi = \cos^{-1}\left(\frac{-3}{5}\right)$

$\phi \approx 2.21$

$\phi \approx 127^\circ$

Free Damped Motion

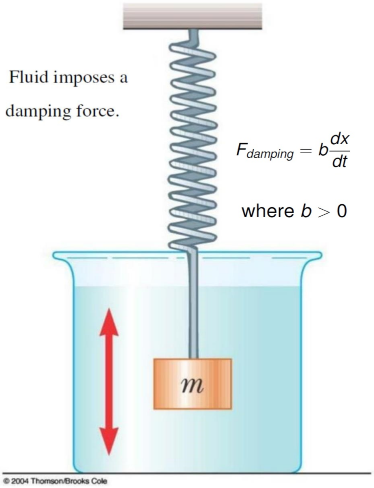


Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of spring + Force of damping

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{b}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

2 distinct real roots.

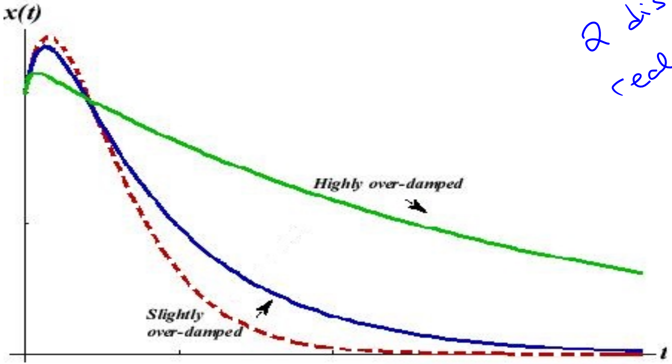


Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

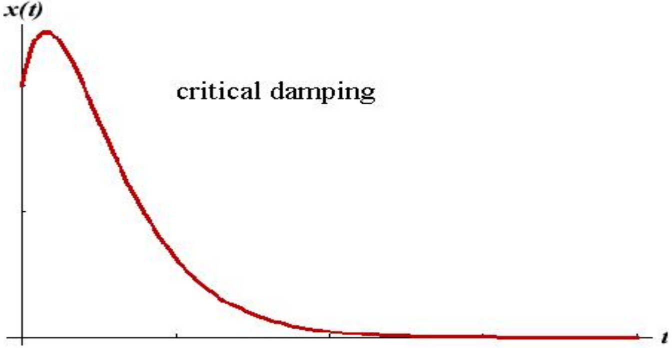


Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1\cos(\omega_1 t) + c_2\sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

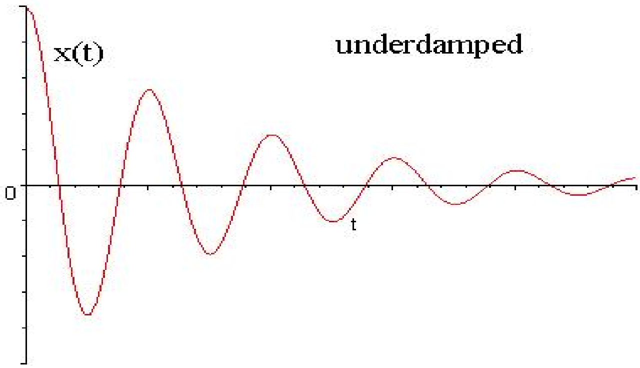
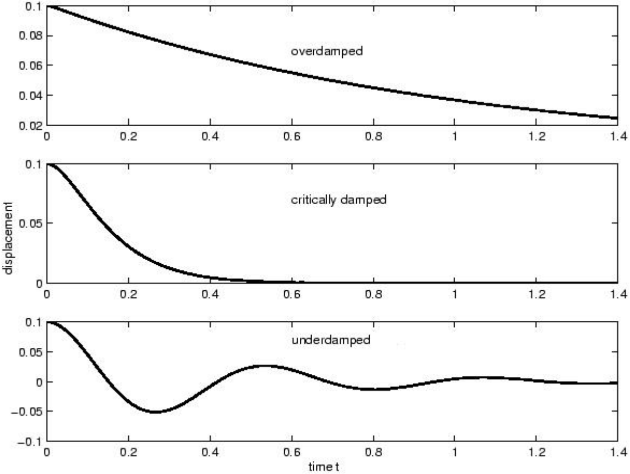


Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping



2 real

repeated real

complex roots

Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The model is

$$mx'' + bx' + kx = 0$$

Here, $m = 2$ kg, $k = 12$ N/m, $b = 10$

Our ODE is

$$2x'' + 10x' + 12x = 0.$$

In standard form,

$$x'' + 5x' + 6x = 0$$

The characteristic eqn is

$$r^2 + 5r + 6 = 0$$

which factors as $(r+2)(r+3) = 0$

$$r = -2 \quad \text{or} \quad r = -3$$

we have two distinct real roots,
so the system is over damped.

The ODE is $x'' + 5x' + 6x = 0$

so $2\lambda = 5$, and $\omega^2 = 6$

so $\lambda = \frac{5}{2}$ and

$$\lambda^2 - \omega^2 = \left(\frac{5}{2}\right)^2 - 6 = \frac{25}{4} - \frac{24}{4} = \frac{1}{4} > 0.$$

Again, the system is overdamped.

Note, we didn't have to do this second test.
But it shows that our conclusion based on
the roots matches the condition related
to the parameters, lambda and omega.