October 21 Math 2306 sec. 53 Fall 2024

Section 11: Linear Mechanical Equations

We were considering the displacement from equilibrium, $x(t)$, of an object of mass m suspended from a flexible spring with *spring constant* k. In the absence of any sort of damping or external forces, the object exhibits **simple harmonic motion.**

The displacement is subject to the second order, linear, homogeneous differential equation

$$mx'' + kx = 0 \quad \text{i.e.,} \quad x'' + \omega^2 x = 0,$$

where the parameter

$$\omega^2 = \frac{k}{m}.$$

Equilibrium & Displacment in Equilibrium

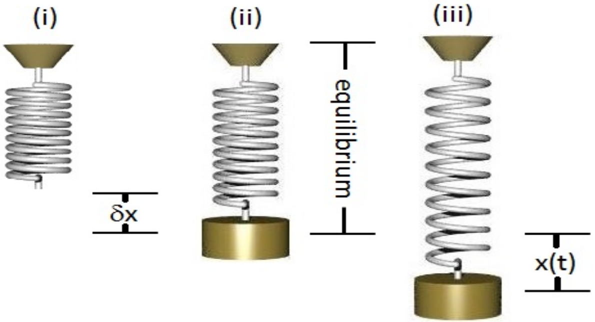


Figure: Hooke's law states that the displacement in equilibrium, δx, is related to the object's weight, W, via $W = k\delta x$. We'll use the convention

$$x > 0 \quad \text{above equilibrium, and} \quad x < 0 \quad \text{below equilibrium.}$$

Remark: Since $W = mg = k\delta x$, $\omega^2 = \dfrac{k}{m} = \dfrac{g}{\delta x}$.

Simple Harmonic Motion

With initial displacement x_0 and initial velocity, x_1, the position $x(t)$ at time t satisfies the IVP

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1.$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[1]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[1]Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi) = A \cos(\omega t - \hat{\phi})$$

where the amplitude

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ or $\hat{\phi}$ must be defined by

$$\sin \phi = \frac{x_0}{A}, \text{ and } \cos \phi = \frac{x_1}{\omega A} \quad \text{or} \quad \cos \hat{\phi} = \frac{x_0}{A}, \text{ and } \sin \hat{\phi} = \frac{x_1}{\omega A}.$$

Note, $\phi + \hat{\phi} = \frac{\pi}{2}$ (up to an integer multiple of 2π).

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion in the form $x = A\sin(\omega t + \phi)$, and identify the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

From the weight $W = 4$lb and displacement in equilibrium $\delta x = \frac{1}{2}$ ft, we found the mass and spring constant

$$m = \frac{1}{8}\text{ slugs}, \quad \text{and} \quad k = 8\,\frac{\text{lb}}{\text{ft}}.$$

This makes

$$\omega^2 = 64\,\frac{1}{\text{sec}^2}.$$

We ended up with the ODE

$$\frac{1}{8}x'' + 8x = 0 \quad \Longrightarrow \quad x'' + 64x = 0.$$

We need to identify $x(0)$ and $x'(0)$.

$x(0) = 4$ (4ft above equilibrium)

$x'(0) = -24$ (24 ft/sec downward)

The characteristic equation is

$$r^2 + 64 = 0 \Rightarrow r^2 = -64, \quad r = \pm 8i$$

$$x(t) = c_1 \cos(8t) + c_2 \sin(8t)$$

$$x'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t)$$

$$x(0) = c_1 \cos(0) + c_2 \sin(0) = 4 \Rightarrow c_1 = 4$$

$$x'(0) = -8C_1\sin(0) + 8C_2\cos(0) = -24$$

$$8C_2 = -24 \Rightarrow C_2 = -3$$

The displacement

$$x(t) = 4\cos(8t) - 3\sin(8t)$$

Let $A = \sqrt{4^2 + (-3)^2} = 5$. This is the amplitude.

$$x(t) = 5\sin(8t + \phi)$$

where $\sin\phi = \frac{4}{5}$ and $\cos\phi = \frac{-3}{5}$

The amplitude $A = 5$.

The period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

Linear frequency $f = \frac{4}{\pi}$

The phase shift $\phi = \cos^{-1}\left(-\frac{3}{5}\right)$.

$$\phi \approx 2.21$$

$$\approx 127^\circ$$

Free Damped Motion

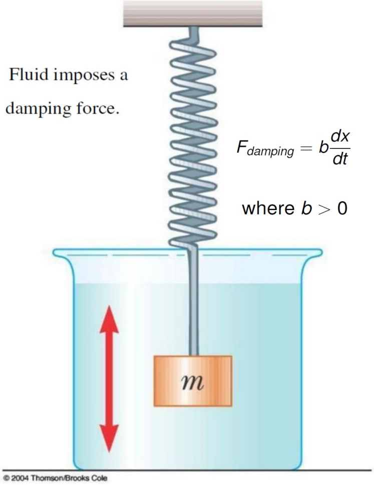


Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of spring + Force of damping

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{b}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

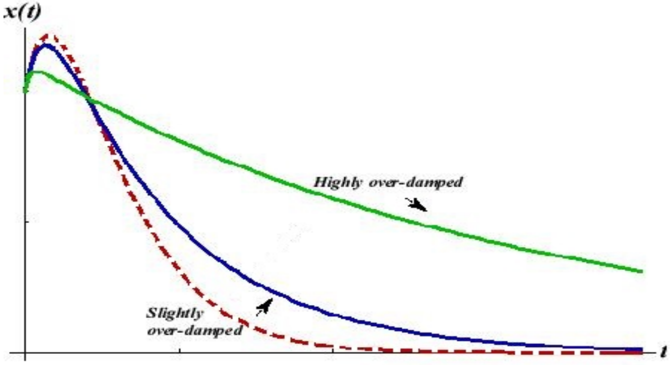


Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

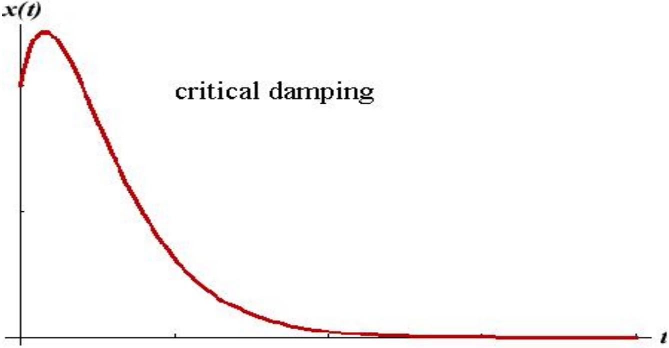


Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1\cos(\omega_1 t) + c_2\sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

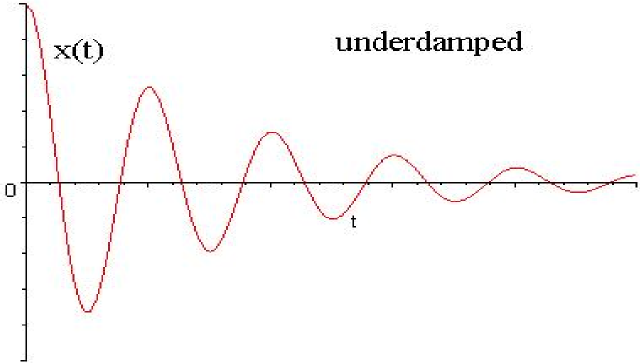
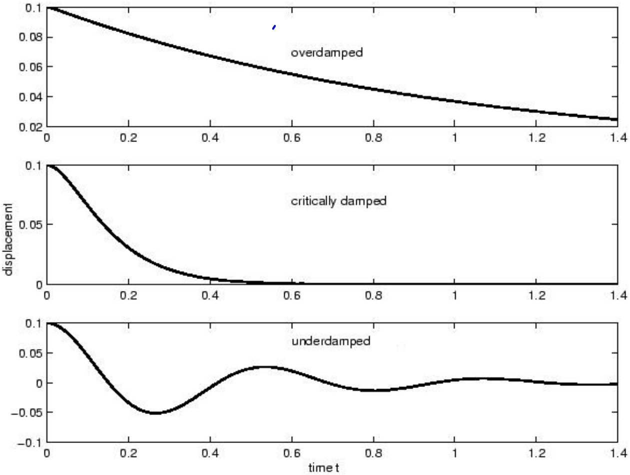


Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping



2 distinct real roots

1 repeated real root

Complex roots

Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The model is

$$mx'' + bx' + kx = 0$$

We need m, b and k.

Here, $m = 2$, $b = 10$, $k = 12$.

The ODE is

$$2x'' + 10x' + 12x = 0$$

In standard form, we have

$$x'' + 5x' + 6x = 0$$

The characteristic equation is

$$r^2 + 5r + 6 = 0$$

which factors, $(r + 2)(r + 3) = 0$

$r = -2$ or $r = -3$.

Two distinct real roots $\Rightarrow$ the system is <u>over damped</u>.

The system is overdamped if $\lambda^2 > \omega^2$

$$x'' + 5x' + 6x = 0$$

$$x'' + 2\lambda x' + \omega^2 x = 0 \Rightarrow$$

$\omega^2 = 6$

$2\lambda = 5$

$\lambda = 5/2$

$$\lambda^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}, \quad \omega^2 = 6 = \frac{24}{4}$$

$$\lambda^2 > \omega^2$$