October 22 Math 2306 sec. 52 Fall 2021

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$?

By definition $\mathscr{L}\left\{e^t t^2\right\}=\int_0^\infty e^{-st}e^t t^2\,dt$

$e^{-st}\,e^{t}$

$= e^{-st+t}$

$= e^{-(s-1)t}$

$$= \int_0^\infty e^{-(s-1)t}\, t^2\, dt$$

$$= \int_0^\infty e^{-pt}\, t^2\, dt \quad \text{if } p = s-1$$

$$= F(p) \quad \text{where} \quad F(s) = \mathscr{L}\{t^2\}$$

$\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$

If $f(t)=t^2$, then $F(s)=\mathscr{L}\left\{t^2\right\}=\dfrac{2}{s^3}$, and

$$F(s-1)=\frac{2}{(s-1)^3}=\mathscr{L}\left\{e^t t^2\right\}.$$

In general, if $F(s)=\mathscr{L}\{f(t)\}$, then

$$F(s-a)=\int_0^\infty e^{-(s-a)t}f(t)\,dt=\int_0^\infty e^{-st}\left[e^{at}f(t)\right]dt$$

$$=\mathscr{L}\left\{e^{at}f(t)\right\}$$

Theorem (translation in s)

Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

In other words, if $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

Example

Evaluate

(a) $\mathscr{L}\{t^6 e^{3t}\} = \dfrac{6!}{(s-3)^7}$

a) $\mathscr{L}\{t^6\} = \dfrac{6!}{s^7} = F(s)$

$a = 3$

(b) $\mathscr{L}\{e^{-t}\cos(t)\} = \dfrac{s-(-1)}{(s-(-1))^2+1}$

$= \dfrac{s+1}{(s+1)^2+1}$

b) $\mathscr{L}\{\cos t\} = \dfrac{s}{s^2+1} = F(s)$

$a = -1$

(c) $\mathscr{L}\{e^{-t}\sin(t)\}$

$= \dfrac{1}{(s+1)^2+1}$

c) $\mathscr{L}\{\sin t\} = \dfrac{1}{s^2+1} = F(s)$

$a = -1$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

Can we do partial fractions on $\dfrac{s}{s^2+2s+2}$

$b^2-4ac = 2^2-4\cdot1\cdot2 = -4 < 0$

s^2+2s+2 is irreducible $\Rightarrow$ complete the square.

Complete the square

$s^2+2s+2 = s^2+2s+1-1+2 = (s+1)^2+1$

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1}$$

we need $s+1$ here too.

Use that $s = s+1-1$

Hence

$$\frac{s}{s^2+2s+2} = \frac{s+1-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

looks like $\frac{s}{s^2+1}$ $\qquad \frac{1}{s^2+1}$

If $s-a = s+1$ then $a=-1$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

We'll do a partial fraction decomp.

$$\frac{1+3s-s^2}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Clear fractions

$$1+3s-s^2 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$-s^2 + 3s + 1 = (A+B)s^2 + (-2A-B+C)s + A$$

$$A+B=-1$$

$$-2A-B+C=3$$

$$A=1 \qquad B=-1-A=-2$$

$$C=3+2A+B=3+2-2=3$$

$$\frac{1+3s-s^2}{s(s-1)^2}=\frac{1}{s}-\frac{2}{s-1}+\frac{3}{(s-1)^2}$$

looks like $3\frac{1}{s^2}$

$$\mathcal{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3e^{t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 1 - 2e^{t} + 3e^{t}\,t$$