

4.3 Bases

Definition of a Basis

Let S be a subspace of R^n , and let $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_k\}$ be a subset of vectors in S . $\mathcal{B}$ is a **basis** of S provided

- ▶ $\mathcal{B}$ spans S , and
- ▶ $\mathcal{B}$ is linearly independent.

A **basis** is a **linearly independent** spanning set. We can think of a basis as a minimal spanning set. Every vector in the basis contributes something to the span, and it's not possible to remove any vectors from the basis without losing part of the span.

Standard a.k.a. Elementary Basis of R^n

The set $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ of standard unit vectors in R^n is called the **standard basis** or the **elementary basis** of R^n .

For example,

$$R^2 = \text{Span}\{\vec{e}_1, \vec{e}_2\},$$

$$R^3 = \text{Span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\},$$

and so forth.

Elementary bases are easy to work with, but they're not the only bases we can work with.

Recall

Back in chapter 1^a, we showed that every vector $\vec{x} = \langle x_1, x_2 \rangle$ in $\mathbb{R}^2$ could be written as a linear combination

$$\vec{x} = \left(\frac{x_1 + x_2}{2} \right) \langle 1, 1 \rangle + \left(\frac{x_1 - x_2}{2} \right) \langle 1, -1 \rangle.$$

^aSee slides from August 29.

Example: Show that $\{\langle 1, 1 \rangle, \langle 1, -1 \rangle\}$ is a basis for $\mathbb{R}^2$.

We have to show that this set spans $\mathbb{R}^2$ and is linearly independent.

The chapter 1 example shows that
 $\text{Span} \{ \langle 1, 1 \rangle, \langle 1, -1 \rangle \} = \mathbb{R}^2$

Since every vector in $\mathbb{R}^2$ can be written as a linear combo of $\langle 1, 1 \rangle$ and $\langle 1, -1 \rangle$.

We still have to show linear independence.

Let $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and consider $B\vec{x} = \vec{0}_2$.

$$[B \mid \vec{0}_2] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

If $\vec{x} = \langle x_1, x_2 \rangle$ then $x_1 = 0$ and $x_2 = 0$.

The columns of B are linearly independent. So $\{\langle 1, 1 \rangle, \langle 1, -1 \rangle\}$ is a basis for $\mathbb{R}^2$.

Note: We do have to show that $\text{Span}\{\langle 1, 1 \rangle, \langle 1, -1 \rangle\} = \mathbb{R}^2$. But that was already done in the example in Chapter 1 (see the purple box above). We're just not repeating that work here because it was already done.

Example

Determine whether the set $\{\langle 1, 0, 0 \rangle, \langle 1, 1, 0 \rangle, \langle 1, 1, 1 \rangle, \langle 0, 1, 0 \rangle\}$ is a basis for $\mathbb{R}^3$.

We have to determine if the set spans $\mathbb{R}^3$ and if it is linearly independent.

It's not lin. independent since there are 4 vectors in $\mathbb{R}^3$. So it's not a basis.

Basis for a Null Space

Find a basis for $\mathcal{N}(A)$ for $A = \begin{bmatrix} -2 & -5 & 3 & -3 \\ 4 & 8 & 0 & 4 \\ -5 & -6 & -12 & -1 \end{bmatrix}$.

If $\vec{x} \in \mathcal{N}(A)$ then

$A\vec{x} = \vec{0}_3$. Note that $\vec{x} \in \mathbb{R}^4$.

$$[A \mid \vec{0}_3] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & 6 & -1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} x_1 = -6x_3 + x_4 \\ x_2 = 3x_3 - x_4 \\ x_3, x_4 \text{ - free} \end{array}$$

$$\vec{x} = \langle x_1, x_2, x_3, x_4 \rangle = \langle -6x_3 + x_4, 3x_3 - x_4, x_3, x_4 \rangle$$

$$\text{rref}(A) = \left[\begin{array}{cccc} 1 & 0 & 6 & -1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] = \langle -6x_3, 3x_3, x_3, 0 \rangle + \langle x_4, -x_4, 0, x_4 \rangle$$

$$\vec{x} = x_3 \langle -6, 3, 1, 0 \rangle + x_4 \langle 1, -1, 0, 1 \rangle$$

$\{ \langle -6, 3, 1, 0 \rangle, \langle 1, -1, 0, 1 \rangle \}$ spans $N(A)$.

Consider $c_1 \langle -6, 3, 1, 0 \rangle + c_2 \langle 1, -1, 0, 1 \rangle = \langle 0, 0, 0, 0 \rangle$

$$\langle -6c_1 + c_2, 3c_1 - c_2, c_1, c_2 \rangle = \langle 0, 0, 0, 0 \rangle.$$

From the third and fourth entries, $c_1 = c_2 = 0$.

So $\{ \langle -6, 3, 1, 0 \rangle, \langle 1, -1, 0, 1 \rangle \}$ is lin.
independent. This set is a basis for

$N(A)$.

Basis for $\mathcal{N}(A)$

If $\mathcal{N}(A) \neq \{\vec{0}_n\}$, then our process^a for finding a spanning set for $\mathcal{N}(A)$ produces a basis for $\mathcal{N}(A)$.

^aExpressing basic variables in terms of free variables, and decomposing the vectors to separate free variables.

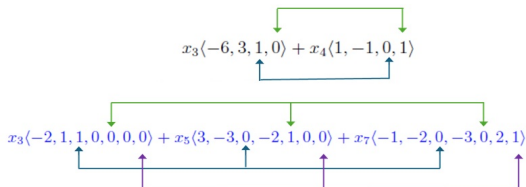


Figure: The vectors in the decomp will have a distribution of 1s and 0s that result in their being linearly independent. If you set the linear combination equal to the zero vector, every coefficient (x_3 and x_4 on top and x_3 , x_5 and x_7 for the bottom) will have to be zero.

Why are Bases Special?

Coordinate Vectors

Theorem

Let $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k\}$ be an ordered basis of a subspace S of $\mathbb{R}^n$. If $\vec{x}$ is any element of S , then there is exactly one representation (i.e., one set of coefficients) of $\vec{x}$ as a linear combination of elements of $\mathcal{B}$.

Suppose $\vec{x}$ in S can be written in two ways

$$\vec{x} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_k \vec{u}_k \quad \text{and}$$

$$\vec{x} = a_1 \vec{u}_1 + a_2 \vec{u}_2 + \dots + a_k \vec{u}_k.$$

Subtract one line from the other.

$$\vec{x} - \vec{x} = (c_1 - a_1) \vec{u}_1 + (c_2 - a_2) \vec{u}_2 + \dots + (c_k - a_k) \vec{u}_k$$

Note: Saying the basis is **ordered** just means that we put them in a particular order and number them accordingly.

$$\vec{0}_n = (c_1 - a_1)\vec{u}_1 + (c_2 - a_2)\vec{u}_2 + \cdots + (c_k - a_k)\vec{u}_k.$$

The $\vec{u}$'s are lin. independent. This has only the trivial solution.

$$c_1 - a_1 = 0 \Rightarrow c_1 = a_1$$

$$c_2 - a_2 = 0 \Rightarrow c_2 = a_2$$

$$\vdots$$

$$c_k - a_k = 0 \Rightarrow c_k = a_k$$

Definition: Coordinate Vectors

Let S be a subspace of R^n and $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_k\}$ be an ordered basis of S . For each element $\vec{x}$ in S , the **coordinate vector for $\vec{x}$ relative to the basis $\mathcal{B}$** is denoted $[\vec{x}]_{\mathcal{B}}$ and is defined to be

$$[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2, \dots, c_k \rangle,$$

where the entries are the coefficients of the representation of $\vec{x}$ as a linear combination of the basis elements. That is, the c 's are the coefficients in the equation

$$\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k.$$

Example

Consider the basis $\mathcal{B} = \{\langle 2, 1 \rangle, \langle -1, 1 \rangle\}$, in the order given, of $\mathbb{R}^2$. Determine

1. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle 2, 1 \rangle$
2. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle -1, 1 \rangle$
3. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle 1, 0 \rangle$
4. $\vec{x}$ if $[\vec{x}]_{\mathcal{B}} = \langle -1, -1 \rangle$

Let $\vec{b}_1 = \langle 2, 1 \rangle$ and
 $\vec{b}_2 = \langle -1, 1 \rangle$.

1. $[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2 \rangle$ where
 $\langle 2, 1 \rangle = c_1 \vec{b}_1 + c_2 \vec{b}_2 = c_1 \langle 2, 1 \rangle + c_2 \langle -1, 1 \rangle$
 $c_1 = 1$ and $c_2 = 0$

$$[\vec{x}]_{\mathcal{B}} = \langle 1, 0 \rangle.$$

2. $[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2 \rangle$ $\langle -1, 1 \rangle = c_1 \langle 2, 1 \rangle + c_2 \langle -1, 1 \rangle$

$$B = \{\langle 2, 1 \rangle, \langle -1, 1 \rangle\} \quad 2. [\vec{x}]_B = \langle 0, 1 \rangle.$$

In terms of the basis $\{\vec{b}_1, \vec{b}_2\}$ of $\mathbb{R}^2$, the vectors $\vec{b}_1$ and $\vec{b}_2$ are standard unit vectors.

$$[\vec{b}_1]_B = \langle 1, 0 \rangle \text{ and}$$

$$[\vec{b}_2]_B = \langle 0, 1 \rangle.$$

We'll finish this exercise next time.