October 22 Math 3260 sec. 53 Fall 2025

4.3 Bases

Definition of a Basis

Let S be a subspace of R^n, and let $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ be a subset of vectors in S. $\mathcal{B}$ is a **basis** of S provided

- $\mathcal{B}$ spans S, and
- $\mathcal{B}$ is linearly independent.

A **basis** is a **linearly independent** spanning set. We can think of a basis as a minimal spanning set. Every vector in the basis contributes something to the span, and it's not possible to remove any vectors from the basis without losing part of the span.

Standard a.k.a. Elementary Basis of R^n

The set $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of standard unit vectors in R^n is called the **standard basis** or the **elementary basis** of R^n.

For example,

$$\begin{aligned} R^2 &= \text{Span}\{\vec{e}_1, \vec{e}_2\}, \\ R^3 &= \text{Span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}, \end{aligned}$$

and so forth.

Elementary bases are easy to work with, but they're not the only bases we can work with.

Recall

Back in chapter 1[a], we showed that every vector $\vec{x} = \langle x_1, x_2 \rangle$ in R^2 could be written as a linear combination

$$\vec{x} = \left(\frac{x_1 + x_2}{2}\right)\langle 1, 1\rangle + \left(\frac{x_1 - x_2}{2}\right)\langle 1, -1\rangle.$$

[a] See slides from August 29.

Example: Show that $\{\langle 1, 1\rangle, \langle 1, -1\rangle\}$ is a basis for R^2.

We have to show that this set spans R^2 and is linearly independent.

From the Chapter 1 example, every vector $\vec{x} = \langle x_1, x_2 \rangle$ in R^2 can be written as

$$\langle x_1, x_2\rangle = c_1\langle 1,1\rangle + c_2\langle 1,-1\rangle.$$

So $\text{Span}\{\langle 1,1\rangle, \langle 1,-1\rangle\} = \mathbb{R}^2$.

To show that the set is linearly independent, let $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Consider

$B\vec{x} = \vec{0}_2$.

$$[B \mid \vec{0}_2] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$B\vec{x} = \vec{0}_2$ has only the trivial solution,

so the columns of B are lin.

independent.

That is, $\{\langle 1,1\rangle, \langle 1,-1\rangle\}$ is a

linearly independent set that spans $\mathbb{R}^2$.

It is a basis of $\mathbb{R}^2$.

Example

Determine whether the set $\{\langle 1,0,0\rangle, \langle 1,1,0\rangle, \langle 1,1,1\rangle, \langle 0,1,0\rangle\}$ is a basis for R^3.

We need to determine whether the set spans $\mathbb{R}^3$ and whether it is lin. independent.

The set is linearly dependent.

$$\langle 1,1,0\rangle + (-1)\langle 1,0,0\rangle = \langle 0,1,0\rangle.$$

It is not a basis of $\mathbb{R}^3$.

Basis for a Null Space

Find a basis for $\mathcal{N}(A)$ for $A = \begin{bmatrix} -2 & -5 & 3 & -3 \\ 4 & 8 & 0 & 4 \\ -5 & -6 & -12 & -1 \end{bmatrix}$.

If $\vec{x} \in \mathcal{N}(A)$, then

$A\vec{x} = \vec{0}_3$. solve $A\vec{x} = \vec{0}_3$,

for $\vec{x} = \langle x_1, x_2, x_3, x_4 \rangle$

$$[A \mid \vec{0}_3] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & 6 & -1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -6x_3 + x_4$

$x_2 = 3x_3 - x_4$

x_3, x_4 are free

$$rref(A) = \begin{bmatrix} 1 & 0 & 6 & -1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{x} = \langle -6x_3 + x_4, 3x_3 - x_4, x_3, x_4 \rangle$

$$\vec{x} = \langle -6x_3, 3x_3, x_3, 0\rangle + \langle x_4, -x_4, 0, x_4\rangle$$

$$= x_3\langle -6, 3, 1, 0\rangle + x_4\langle 1, -1, 0, 1\rangle$$

A spanning set for $N(A)$ is $\{\langle -6,3,1,0\rangle, \langle 1,-1,0,1\rangle\}$.

To show linear independence, consider

$$c_1\langle -6,3,1,0\rangle + c_2\langle 1,-1,0,1\rangle = \langle 0,0,0,0\rangle$$

$$\langle -6c_1+c_2, 3c_1-c_2, c_1, c_2\rangle = \langle 0,0,0,0\rangle$$

$c_1 = 0$ and $c_2 = 0$ (only the trivial solution)

So $\{\langle -6,3,1,0\rangle, \langle 1,-1,0,1\rangle\}$ is linearly independent, making it a basis of $N(A)$.

Basis for $\mathcal{N}(A)$

If $\mathcal{N}(A) \neq \{\vec{0}_n\}$, then our process[a] for finding a spanning set for $\mathcal{N}(A)$ produces a basis for $\mathcal{N}(A)$.

[a] Expressing basic variables in terms of free variables, and decomposing the vectors to separate free variables.

$$x_3\langle -6, 3, 1, 0\rangle + x_4\langle 1, -1, 0, 1\rangle$$

$$x_3\langle -2, 1, 1, 0, 0, 0, 0\rangle + x_5\langle 3, -3, 0, -2, 1, 0, 0\rangle + x_7\langle -1, -2, 0, -3, 0, 2, 1\rangle$$

Figure: The vectors in the decomp will have a distribution of 1s and 0s that result in their being linearly independent. If you set the linear combination equal to the zero vector, every coefficient (x_3 and x_4 on top and x_3, x_5 and x_7 for the bottom) will have to be zero.

Why are Bases Special?

Coordinate Vectors

Theorem

Let $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ be an ordered basis of a subspace S of R^n. If $\vec{x}$ is any element of S, then there is exactly one representation (i.e., one set of coefficients) of $\vec{x}$ as a linear combination of elements of $\mathcal{B}$.

Suppose $\vec{x} \in S$ has two representations,

$$\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k \quad \text{and}$$

$$\vec{x} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_k\vec{u}_k.$$

Subtract the bottom line from the top.

$$\vec{x} - \vec{x} = (c_1-a_1)\vec{u}_1 + (c_2-a_2)\vec{u}_2 + \cdots + (c_k-a_k)\vec{u}_k$$

Note: Saying the basis is **ordered** just means that we put them in a particular order and number them accordingly.

$\vec{0}_n = (c_1 - a_1)\vec{u}_1 + (c_2 - a_2)\vec{u}_2 + \cdots + (c_k - a_k)\vec{u}_k.$

By linear independence all coefficients must be zero.

$$c_1 - a_1 = 0 \Rightarrow c_1 = a_1$$
$$c_2 - a_2 = 0 \Rightarrow c_2 = a_2$$
$$\vdots$$
$$c_k - a_k = 0 \Rightarrow c_k = a_k$$

Definition: Coordinate Vectors

Let S be a subspace of R^n and $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ be an ordered basis of S. For each element $\vec{x}$ in S, the **coordinate vector for $\vec{x}$ relative to the basis $\mathcal{B}$** is denoted $[\vec{x}]_{\mathcal{B}}$ and is defined to be

$$[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2, \ldots, c_k \rangle,$$

where the entries are the coefficients of the representation of $\vec{x}$ as a linear combination of the basis elements. That is, the c's are the coefficients in the equation

$$\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k.$$

Example

Consider the basis $\mathcal{B} = \{\langle 2,1\rangle, \langle -1,1\rangle\}$, in the order given, of R^2. Determine

1. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle 2,1\rangle$
2. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle -1,1\rangle$
3. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle 1,0\rangle$
4. $\vec{x}$ if $[\vec{x}]_{\mathcal{B}} = \langle -1,-1\rangle$

$\vec{b}_1 = \langle 2,1\rangle, \quad \vec{b}_2 = \langle -1,1\rangle$

1. $[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2\rangle$ where $\langle 2,1\rangle = c_1\vec{b}_1 + c_2\vec{b}_2$

$\langle 2,1\rangle = c_1\langle 2,1\rangle + c_2\langle -1,1\rangle$

$c_1 = 1, \quad c_2 = 0$

$[\vec{x}]_{\mathcal{B}} = \langle 1,0\rangle$

2. $[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2\rangle$ $\quad \langle -1,1\rangle = c_1\langle 2,1\rangle + c_2\langle -1,1\rangle$

$= \langle 0,1\rangle.$

$\mathcal{B} = \{\langle 2, 1\rangle, \langle -1, 1\rangle\}$

In terms of this basis, the basis vectors look like standard unit vectors,

$$[\vec{b}_1]_{\mathcal{B}} = \langle 1, 0\rangle \text{ and}$$

$$[\vec{b}_2]_{\mathcal{B}} = \langle 0, 1\rangle.$$

We will finish this exercise next time.