October ~~20~~ 23 Math 2306 sec. 52 Spring 2023

Section ~~13~~ 11: Linear Mechanical Equations

Free Motion

If an object of mass m is attached to a flexible spring with spring stiffness k, then the spring force associated with a displacement x (from equilibrium) is given by Hooke's Law

$$F_{spring} = kx.$$

If the object is subjected to linear damping (e.g., due to friction or a dashpot) with damping coefficient b, then the damping force is

$$F_{damping} = b\frac{dx}{dt}.$$

Free Motion

In the absence of any external forces, the displacement satisfies

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

The corresponding motion is

- undamped if $b = 0$,
- over damped if $b^2 - 4mk > 0$,
- critically damped if $b^2 - 4mk = 0$, and
- under damped if $b^2 - 4mk < 0$.

> - Over damping corresponds to two distinct real roots.
> - Critical damping corresponds to a repeated root (perfect square quadratic).
> - Under damping corresponds to complex roots (with nonzero real part).
>
> If the roots are complex with zero real part, the system is **undamped**, i.e., $b = 0$.

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + f(t), \quad b \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t).$$

$$\text{where} \quad 2\lambda = \frac{b}{m}, \quad \omega^2 = \frac{k}{m}.$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and} \quad (2) \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

suppose $\gamma \neq \omega$

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$

This is correct, it has no like terms in common w/ x_c. The general solution looks like

$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + A\cos(\gamma t) + B\sin(\gamma t)$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$$x_p = A\cos(\gamma t) + B\sin(\gamma t)$$

$= A \cos(\omega t) + B \sin(\omega t)$

Suppose $\gamma = \omega$.

Not correct as it duplicates x_c.

The correct form is $x_p (A \cos(\omega t) + B \sin(\omega t)) t$

The general solution is

$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + A t \cos(\omega t) + B t \sin(\omega t)$.

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2}\left(\sin(\gamma t) - \frac{\gamma}{\omega}\sin(\omega t)\right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

Case (2): $x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t\cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.