October 23 Math 2306 sec. 53 Fall 2024

Section 11: Linear Mechanical Equations

We are considering a flexible spring with an object attached. In the absence of damping and driving, we get **simple harmonic motion**.

$$mx'' + kx = 0$$

With linear damping and no driving, we get a spring-mass-damping system.

$$mx'' + bx' + kx = 0$$

- m is mass (in kg or slugs),
- b is the damping coefficient (in N/(m/sec) or lbs/(ft/sec))
- k is the spring constant (in N/m or lb/ft)
- $x(t)$ is the position/displacement from equilibrium (in m or ft) at time t in seconds.

Damping Types

For the damped motion, there are three damping types that equate to the three types of roots of the characteristic equation.

- Over damping (two distinct real roots),
- Critical damping (one repeated real root),
- Under damping (complex conjugate roots)

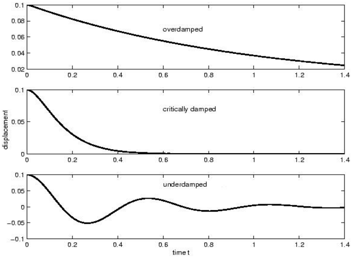

Figure: Comparison of motion for the three damping types.

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

The model is $mx'' + bx' + kx = 0$. We need m, b, and k. Here,

$m = 3$ kg, k $12 \frac{N}{m}$, $b = 12 \frac{N}{m/sec}$

The ODE is

$$3x'' + 12x' + 12x = 0$$

In Standard form

$$x'' + 4x' + 4x = 0$$

The characteristic equation is

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \Rightarrow r = -2 \quad \text{Double root}$$

The system is <u>critically damped</u>.

The initial conditions are

$x(0) = 0$ (at equilibrium)

$x'(0) = 1$ (upward 1 m/sec)

From $r = \cdot\ \cdot$, $x_1 = e^{-2t}$ and $x_2 = t e^{-2t}$

And $X(t) = c_1 e^{-2t} + c_2 t e^{-2t}$.

Apply the I.C.

$$x'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$X(0) = c_1 e^{0} + c_2 (0) e^{0} = 0 \Rightarrow c_1 = 0$$

$$x'(0) = -2c_1 e^{0} + c_2 e^{0} - 2c_2 (0) e^{0} = 1$$

$$-2c_1 + c_2 = 1 \Rightarrow c_2 = 1$$

The position

$$x(t) = t\,e^{-2t}$$

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + f(t), \quad b \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$mx'' + bx' + kx = f(t)$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

$\omega^2 = \frac{k}{m}$ and $2\lambda = \frac{b}{m}$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and } (2) \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

Suppose $\gamma \neq \omega$

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$

Since $\gamma \neq \omega$, this has no terms in common w/ x_c, so it is correct.

The solution would be

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + A\cos(\gamma t) + B\sin(\gamma t)$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

Suppose $\gamma = \omega$.

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$

$= A\cos(\omega t) + B\sin(\omega t)$

This shares like terms w/ x_c.

$x_p = (A\cos(\omega t) + B\sin(\omega t))t = At\cos(\omega t) + Bt\sin(\omega t)$

This is correct. The solution is

$x = c_1\cos(\omega t) + c_2\sin(\omega t) + At\cos(\omega t) + Bt\sin(\omega t)$

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

from rest

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2} \left(\sin(\gamma t) - \frac{\gamma}{\omega} \sin(\omega t) \right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

Case (2): $x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t \cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.

Example

A 3 kg mass is attached to a spring with spring constant 48 N/m. Assume damping is negligible and the system is driven by a force $f(t) = F_0 \cos(\gamma t)$.

(a) What value of γ would induce pure resonance?

(b) If the object starts from rest from the equilibrium position, find the displacement in the pure resonance case.

The model is $mx'' + bx' + kx = f(t)$

$b = 0$ (no damping)

$m = 3$ kg and $k = 48\ \frac{N}{m}$, $f(t) = F_0 \cos(\gamma t)$

$$3x'' + 48x = F_0 \cos(\gamma t)$$

In standard form,

$$x'' + 16x = \frac{F_0}{3}\cos(\gamma t)$$

$$\omega^2 = \frac{k}{m} = 16 \implies \omega = 4.$$

a) Pure resonance occurs when $\gamma = \omega$,

so the resonance frequency

$$\gamma = 4.$$

b) we have the IVP

$$x'' + 16x = \frac{F_0}{3}\cos(4t),$$

$$x(0) = 0 \quad \text{and} \quad x'(0) = 0$$

$$X_c = C_1 \cos(4t) + C_2 \sin(4t).$$

$$X_p = (A\cos(4t) + B\sin(4t))\, t$$

$$X_p = At\cos(4t) + Bt\sin(4t)$$

$$X_p' = A\cos(4t) + B\sin(4t) - 4At\sin(4t) + 4Bt\cos(4t)$$

$$X_p'' = -8A\sin(4t) + 8B\cos(4t) - 16At\cos(4t) - 16Bt\sin(4t)$$

$$X_p'' + 16X_p = \frac{F_0}{3}\cos(4t)$$

$$-8A\sin(4t) + 8B\cos(4t) = \frac{F_0}{3}\cos(4t)$$

$$8B = \frac{F_0}{3}, \quad -8A = 0$$

$$B = \frac{F_0}{24}, \quad A = 0$$

$$\boxed{x_p = \frac{F_0}{24} t \sin(4t)}$$

We'll finish next time.