October 24 Math 2306 sec. 51 Fall 2022

Section 15: Shift Theorems

We're going to add two theorems that allow us to take Laplace transforms of a wider variety of functions. The first has one row in our table.

Theorem: If $F(s) = \mathscr{L}\{f(t)\}$ and a is any real number, then $F(s-a) = \mathscr{L}\{e^{at}f(t)\}$.

The second has three rows in our table and says:

Theorem: If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then $e^{-as}F(s) = \mathscr{L}\{f(t-a)\mathscr{U}(t-a)\}$, where $\mathscr{U}$ is the unit step function defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}.$$

Shift (or translation) in s.

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$?

Note that by definition

e^{1t}

$$\mathscr{L}\left\{e^t t^2\right\} = \int_0^\infty e^{-st} e^t t^2\,dt$$

$$= \int_0^\infty e^{-(s-1)t}\, t^2\,dt$$

Let $w = s-1$

$$= \int_0^\infty e^{-wt}\, t^2\,dt = \frac{2!}{w^3} = \frac{2!}{(s-1)^3}$$

Note

$$e^{-st}\cdot e^{t} = e^{-st+t}$$

$$= e^{-(s-1)t}$$

$\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$

If $f(t)=t^2$, then $F(s)=\mathscr{L}\left\{t^2\right\}=\dfrac{2}{s^3}$, and

$$F(s-1)=\frac{2}{(s-1)^3}=\mathscr{L}\left\{e^t t^2\right\}.$$

In general, if $F(s)=\mathscr{L}\{f(t)\}$, then

$$F(s-a)=\int_0^\infty e^{-(s-a)t}f(t)\,dt=\int_0^\infty e^{-st}\left[e^{at}f(t)\right]dt$$

$$=\mathscr{L}\left\{e^{at}f(t)\right\}$$

Theorem (translation in s)

Theorem: Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We can state this in terms of the inverse transform. If $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

Examples: Evaluate

(a) $\mathscr{L}\{t^6 e^{3t}\} = \dfrac{6!}{(s-3)^7}$

$\mathscr{L}\{t^6\} = \dfrac{6!}{s^7} = F(s) \qquad a=3 \qquad F(s-3)$

(b) $\mathscr{L}\{e^{-t}\cos(t)\} = \dfrac{s+1}{(s+1)^2+1}$

$\mathscr{L}\{\cos t\} = \dfrac{s}{s^2+1^2} = F(s) \qquad a=-1 \qquad F(s-(-1)) = F(s+1)$

(c) $\mathscr{L}\{e^{-t}\sin(t)\} = \dfrac{1}{(s+1)^2+1}$

$\mathscr{L}\{\sin t\} = \dfrac{1}{s^2+1^2} = F(s) \qquad a=-1 \qquad F(s+1)$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$.

Partial fractions doesn't apply s^2+2s+2 is irreducible. We'll complete the square.

$$s^2+2s+2 = s^2+2s+1-1 \quad +2$$
$$= (s+1)^2+1$$

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1}$$

Note

$s = s+1-1$

$$= \frac{s+1-1}{(s+1)^2+1}$$

$$= \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} = e^{-1t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = e^{-t}\cos t$$