October 24 Math 3260 sec. 51 Fall 2025

4.3.1 Coordinate Vectors

We defined a **basis** of a subspace S of R^n as a **linearly independent** spanning set. An **ordered basis** is simply a basis in which the vectors are given a specific order (first, second, etc.).

Definition: Coordinate Vectors

Let S be a subspace of R^n and $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ be an ordered basis of S. For each element $\vec{x}$ in S, the **coordinate vector for $\vec{x}$ relative to the basis** $\mathcal{B}$ is denoted $[\vec{x}]_{\mathcal{B}}$ and is defined to be

$$[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2, \ldots, c_k \rangle,$$

where the entries are the coefficients of the representation of $\vec{x}$ as a linear combination of the basis elements. That is, the c's are the coefficients in the equation

$$\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k.$$

Example

Consider the basis $\mathcal{B} = \{\langle 2, 1\rangle, \langle -1, 1\rangle\}$, in the order given, of R^2. Determine

1. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle 2, 1\rangle$
2. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle -1, 1\rangle$
3. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle 1, 0\rangle$
4. $\vec{x}$ if $[\vec{x}]_{\mathcal{B}} = \langle -1, -1\rangle$

Last time, we found that

$$[\langle 2, 1\rangle]_{\mathcal{B}} = \langle 1, 0\rangle \quad \text{because} \quad \langle 2, 1\rangle = 1\langle 2, 1\rangle + 0\langle -1, 1\rangle$$

and

$$[\langle -1, 1\rangle]_{\mathcal{B}} = \langle 0, 1\rangle \quad \text{because} \quad \langle -1, 1\rangle = 0\langle 2, 1\rangle + 1\langle -1, 1\rangle$$

Set $\vec{b}_1 = \langle 2, 1\rangle$, and $\vec{b}_2 = \langle -1, 1\rangle$ and $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$

Determine

3. $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \langle 1, 0\rangle$
4. $\vec{x}$ if $[\vec{x}]_{\mathcal{B}} = \langle -1, -1\rangle$

$[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2\rangle$ where $\langle 1,0\rangle = c_1\vec{b}_1 + c_2\vec{b}_2$

$\langle 1,0\rangle = c_1\langle 2,1\rangle + c_2\langle -1,1\rangle = \langle 2c_1 - c_2, c_1 + c_2\rangle$

$\Rightarrow \begin{array}{l} 2c_1 - c_2 = 1 \\ c_1 + c_2 = 0 \end{array}$ in matrix format $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\langle c_1, c_2\rangle = \langle 1, 0\rangle$

$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{-1}{3} \end{array}\right] \Rightarrow c_1 = \frac{1}{3},\quad c_2 = \frac{-1}{3}$

$[\langle 1,0\rangle]_{\mathcal{B}} = \left\langle \frac{1}{3}, \frac{-1}{3}\right\rangle.$

Set $\vec{b}_1 = \langle 2, 1 \rangle$, and $\vec{b}_2 = \langle -1, 1 \rangle$ and $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$

Check : $\frac{1}{3}\langle 2,1\rangle + \frac{-1}{3}\langle -1,1\rangle = \langle \frac{2}{3}+\frac{1}{3}, \frac{1}{3}-\frac{1}{3}\rangle = \langle 1,0\rangle$

4. $\vec{x}$ if $[\vec{x}]_{\mathcal{B}} = \langle -1, -1 \rangle$

$\vec{x} = -1\, \vec{b}_1 + (-1)\vec{b}_2$

$\vec{x} = -1\,\langle 2,1\rangle + (-1)\,\langle -1,1\rangle$

$= \langle -2, -1\rangle + \langle 1, -1\rangle = \langle -1, -2\rangle$

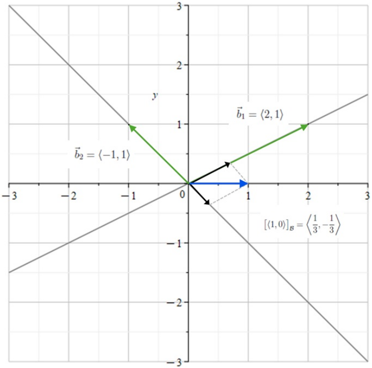

Figure: $[\langle 1, 0 \rangle]_{\mathcal{B}} = \left\langle \frac{1}{3}, -\frac{1}{3} \right\rangle$.

We usually think of this vector as $\langle 1, 0 \rangle = 1\vec{e}_1 + 0\vec{e}_2$. But in this coordinate system, the same vector is

$$\left\langle \frac{1}{3}, -\frac{1}{3} \right\rangle = \frac{1}{3}\vec{b}_1 + \left(-\frac{1}{3}\right)\vec{b}_2$$

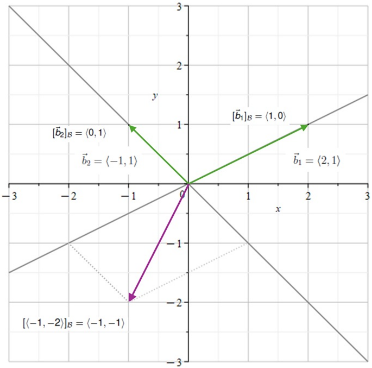

Figure: The coordinate vector $[\vec{x}]_{\mathcal{B}} = \langle -1, -1\rangle$.

This vector is $\langle -1, -1\rangle = -1\ \vec{b}_1 + (-1)\vec{b}_2$. In the standard coordinate system that we are more accustomed to this vector (the pink one) looks like

$$\langle -1, -2\rangle = -1\vec{e}_1 + (-2)\vec{e}_2$$

Change of Basis Matrix

Consider our example $\mathcal{B} = \{\langle 2, 1\rangle, \langle -1, 1\rangle\}$ for R^2. We can create a matrix B having the basis elements as its columns,

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

If $[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2\rangle$ for vector $\vec{x}$, then

$$\vec{x} = \underbrace{c_1\langle 2, 1\rangle + c_2\langle -1, 1\rangle}_{\text{lin. combo of columns}} = B[\vec{x}]_{\mathcal{B}}$$

B is called a **change of basis matrix.**

Change of Basis Matrix

More generally, if we have a basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_k\}$ for the subspace $\mathrm{Span}(\mathcal{B})$ of R^n, then the matrix B having columns

$$\mathrm{Col}_i(B) = \vec{b}_i$$

will be $n \times k$. Then

$$\vec{x} = B[\vec{x}]_{\mathcal{B}}$$

If we know

- $[\vec{x}]_{\mathcal{B}}$, then we use matrix multiplication to find $\vec{x} = B[\vec{x}]_{\mathcal{B}}$,
- $\vec{x}$, then we solve the matrix-vector equation $B[\vec{x}]_{\mathcal{B}} = \vec{x}$ using the augmented matrix $\left[B \,\middle|\, \vec{x}\right]$.

Think of the second case as solving $\underbrace{A\vec{x}}_{B[\vec{x}]_{\mathcal{B}}} = \underbrace{\vec{y}}_{\vec{x}}$

Example

Let $\mathcal{C} = \{\langle 1,1,0\rangle, \langle 0,1,0\rangle\}$ be an ordered basis for the subspace $S = \text{Span}(\mathcal{C})$ of R^3.

Create a matrix C having the basis elements as its columns.

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Use the fact that $\vec{x} = C[\vec{x}]_{\mathcal{C}}$ to find

1. $\vec{x}$ if $[\vec{x}]_{\mathcal{C}} = \langle 4,2\rangle$.

$$\vec{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \langle 4,2\rangle = \langle 1\cdot 4 + 0\cdot 2,\ 1\cdot 4 + 1\cdot 2,\ 0\cdot 4 + 0\cdot 2\rangle$$

$$= \langle 4,6,0\rangle$$

Use the fact that $\vec{x} = C[\vec{x}]_{\mathcal{C}}$ to find

2. $[\vec{u}]_{\mathcal{C}}$ if $\vec{u} = \langle 2, -3, 0\rangle$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} [\vec{u}]_{\mathcal{C}} = \langle 2, -3, 0\rangle$$

$$[C \mid \vec{u}] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{array}\right]$$

$$[\vec{u}]_{\mathcal{C}} = \langle 2, -5\rangle.$$

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Check: $2\langle 1, 1, 0\rangle + (-5)\langle 0, 1, 0\rangle =$

$\langle 2, 2, 0\rangle + \langle 0, -5, 0\rangle = \langle 2, -3, 0\rangle.$

Isomorphic

We might notice that the subspace $S = \mathrm{Span}\{\langle 1, 1, 0\rangle, \langle 0, 1, 0\rangle\}$ in the last example is a subspace of R^3. But we can equate each element uniquely with an element of R^2, namely its coordinate vector. Since we can equate the variable change to matrix multiplication the two operations, vector addition and scalar multiplication, are preserved when working with coordinate vectors. In fact, for every $\vec{u}$ and $\vec{v}$ in S and scalars c and d, it is true that

$$[c\vec{u} + d\vec{v}]_{\mathcal{C}} = c[\vec{u}]_{\mathcal{C}} + d[\vec{v}]_{\mathcal{C}}.$$

There is a name for this property.

We say that S is **isomorphic** to R^2.

`We'll revisit this concept later in chapters five and six.`

4.3.2 Dimension

Theorem:

Suppose S is a subspace of R^n and $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a basis for S that contains k vectors with $k \geq 1$. If $T = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ is any set of m vectors in S with $m > k$, then T is linearly dependent.

Remark: This generalizes the result we had before that a set containing more vectors than elements in each vector must be linearly dependent. It says that a set of vectors having more vectors than elements in a basis for the subspace must be linearly dependent.

Example

1. If S has a basis $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ with three vectors in it, then **every set of vectors in S with four or more vectors in automatically linearly dependent.**

2. If P has a basis $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5\}$ with five vectors in it, then **every set of vectors in P with six or more vectors in automatically linearly dependent.**

Let $\mathcal{B} = \{\langle 1, 2, 0\rangle, \langle 0, 1, 1\rangle\}$ and $S = \mathrm{Span}(\mathcal{B})$. The subset of S,

$$\{\langle 0, 3, 3\rangle, \langle 1, 3, 1\rangle, \langle 2, 5, 1\rangle\}.$$

must be linearly dependent.

Dimension Defined

Theorem

Let $n \geq 2$ and $1 \leq k \leq n$. Suppose S is a subspace of R^n and $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ is a basis of S. Every basis of S consists of exactly k vectors.

Definition: Dimension

Let S be a subspace of R^n. If $S = \{\vec{0}_n\}$, then the dimension of S, written $\dim(S)$ is equal to zero. If S contains more than the zero vector, then the dimension of S, $\dim(S) = k$, where k is the number of elements in any basis of S.

The Dimension of R^n

Note that for $n \geq 2$, $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis for R^n. Hence

$$\dim(R^n) = n.$$

Example: What is the dimension of $\mathcal{N}(A)$ for

$$A = \begin{bmatrix} -2 & -5 & 3 & -3 \\ 4 & 8 & 0 & 4 \\ -5 & -6 & -12 & -1 \end{bmatrix}?$$

For $\vec{x} \in \mathcal{N}(A)$, $A\vec{x} = \vec{0}_3$

$[A \,|\, \vec{0}_3]$

$$[A \,|\, \vec{0}_3] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & 6 & -1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -6x_3 + x_4$

$x_2 = 3x_3 - x_4$

$$\vec{x} = x_3 \langle -6, 3, 1, 0\rangle + x_4 \langle 1, -1, 0, 1\rangle$$

A basis in

$$\{\langle -6, 3, 1, 0\rangle, \langle 1, -1, 0, 1\rangle\}$$

$$\dim(N(A)) = 2$$

Note: The dimension of the null space ends up being the same as the number of free variables. That's no coincidence since we get one basis vector for each free variable.