October 25 Math 2306 sec. 52 Spring 2023

Section 12: LRC Series Circuits

Now that we have solution techniques for second order, linear equations, we return our attention to linear circuits. We can track the charge q on the capacitor, or the current i in an LRC circuit.

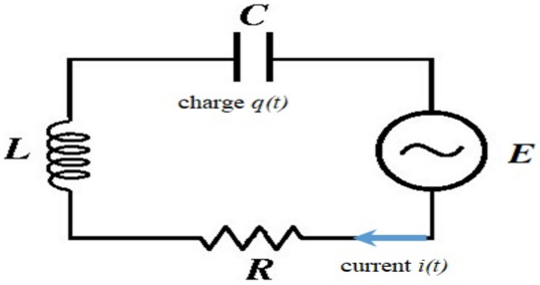

Figure: Simple circuit with inductance L, resistance R, capacitance C, and implied voltage E. The current $i(t) = \frac{dq}{dt}$ where q is the charge on the capacitor at time t in seconds.

Potential Drop Across Each Element

We will recall the voltage drop across each element in terms of charge or current.

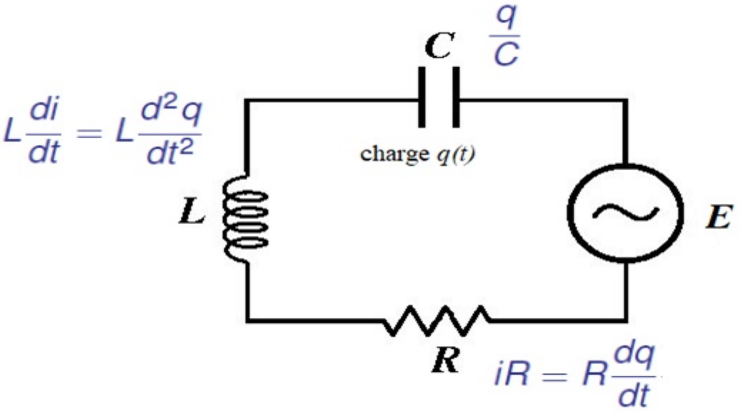

Figure: The potential drop across the capacitor is q/C, across the resistor is iR, and across the inductor is $L\frac{di}{dt}$.

Kirchhoff's Voltage Law

LRC Differential Equation

By Kirchhoff's law, the sum of the potential drops across the passive elements equals the implied voltage. Mathematically, the charge on the capacitor satisfies the second order, linear initial value problem

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0$$

where q_0 and i_0 are the initial charge and current, respectively.

If we take one time derivative, we can get an ODE for the current, $i(t)$:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t)$$

LRC Series Circuit (Free Electrical Vibrations)

Free Electrical Vibrations

If we consider the equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0,$$

the free electrical vibrations are called

$$\begin{array}{lrcl} \textbf{overdamped if} & R^2 - 4L/C & > & 0, \\ \textbf{critically damped if} & R^2 - 4L/C & = & 0, \\ \textbf{underdamped if} & R^2 - 4L/C & < & 0. \end{array}$$

Note that this is the same condition we saw before. Overdamped = two real roots, critically damped = one real root, underdamped = complex roots.

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q.

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

Transient State Charge

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system. The **transient state current** in the circuit $i_c = \frac{dq_c}{dt}$.

Steady and Transient States

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

$$q(t) = q_c(t) + q_p(t).$$

Steady State Charge

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system. The **steady state current** in the circuit $i_p = \frac{dq_p}{dt}$.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

The ODE for charge is $Lq'' + Rq' + \frac{1}{C} q = E$

Here, $L = 0.5$ h, $R = 10\,\Omega$, $C = 4 \cdot 10^{-3}$ f, $E = 5\cos(10t)$

$$\Rightarrow \quad \frac{1}{2} q'' + 10 q' + \frac{1}{4 \cdot 10^{-3}} q = 5\cos(10t)$$

We're asked to determine $i_p = \frac{dq_p}{dt}$

Note $\frac{1}{4 \cdot 10^{-3}} = \frac{10^3}{4} = \frac{1000}{4} = 250$

In standard form, the ODE is

$$q'' + 20q' + 500q = 10\cos(10t)$$

We'll find q_p then find q_p'.

Find q_c: $\quad q_c'' + 20q_c' + 500q_c = 0$

Characteristic eqn $\quad m^2 + 20m + 500 = 0$

Complete the square $\quad m^2 + 20m + 100 = -500 + 100$

$$(m+10)^2 = -400$$

$$m + 10 = \pm\sqrt{-400} = \pm 20i$$

$$m = -10 \pm 20i \qquad \alpha = 10,\ \beta = 20$$

$$q_1 = e^{-10t}\cos(20t), \quad q_2 = e^{-10t}\sin(20t)$$

Find q_p using undetermined coef.

$g(t) = 10\cos(10t)$

$500 \quad g_p = A\cos(10t) + B\sin(10t)$ ✓ correct form

$20 \quad g_p' = -10A\sin(10t) + 10B\cos(10t)$

$1 \quad g_p'' = -100A\cos(10t) - 100B\sin(10t)$

$$g_p'' + 20g_p' + 500g_p = 10\cos(10t)$$

$$\cos(10t)\left(-100A + 200B + 500A\right) + \sin(10t)\left(-100B - 200A + 500B\right) = 10\cos(10t)$$

Matching like terms

$$400A + 200B = 10 \qquad \Rightarrow \qquad 40A + 20B = 1$$

$$-200A + 400B = 0 \qquad\qquad -A + 2B = 0$$

$$\Rightarrow A = 2B$$

$$40(2B) + 20B = 1$$

$$100B = 1 \Rightarrow B = \frac{1}{100}, \quad A = 2B = \frac{2}{100}$$

The steady state charge is

$$q_p = \frac{2}{100}\cos(10t) + \frac{1}{100}\sin(10t)$$

The steady state current $i_p = \frac{dq_p}{dt}$

$$i_p = \frac{-2}{10}\sin(10t) + \frac{1}{10}\cos(10t)$$