October 25 Math 2306 sec. 54 Fall 2021

Section 15: Shift Theorems

Theorem: (translation in s) Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

In other words, if $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

Example: Evaluate

$$\mathscr{L}\left\{e^{4t}\cos(\pi t)\sin(\pi t)\right\}$$

we need to know $\mathscr{L}\{\cos(\pi t)\sin(\pi t)\} = F(s)$.

Recall $\sin 2\theta = 2\sin\theta\cos\theta$

$$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$

$$\cos(\pi t)\sin(\pi t) = \frac{1}{2}\sin(2\pi t)$$

$$F(s) = \mathscr{L}\{\cos(\pi t)\sin(\pi t)\} = \frac{1}{2}\mathscr{L}\{\sin(2\pi t)\}$$

$$= \frac{1}{2}\,\frac{2\pi}{s^2+(2\pi)^2}$$

$\mathcal{L}\{e^{4t}\cos(\pi t)\sin(\pi t)\}$ will be $F(s-4)$.

$$\mathcal{L}\{e^{4t}\cos(\pi t)\sin(\pi t)\} = \frac{\pi}{(s-4)^2+4\pi^2}$$

Evaluate

$$\mathscr{L}^{-1}\left\{\frac{s}{(s+4)^4}\right\}$$

We need to decompose $\frac{s}{(s+4)^4}$.

A partial fraction decomp would have the form

$$\frac{s}{(s+4)^4} = \frac{A}{s+4} + \frac{B}{(s+4)^2} + \frac{C}{(s+4)^3} + \frac{D}{(s+4)^4}$$

Here's a shortcut

$$\frac{s}{(s+4)^4} = \frac{s+4-4}{(s+4)^4} = \frac{s+4}{(s+4)^4} - \frac{4}{(s+4)^4}$$

$$= \frac{1}{(s+4)^3} - \frac{4}{(s+4)^4}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+4)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+4)^3}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^4}\right\}$$

we need $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2!}\,\frac{2!}{s^3}\right\} = \frac{1}{2!}t^2$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3!}\,\frac{3!}{s^4}\right\} = \frac{1}{3!}t^3$$

If $s-a = s+4$, then $a=-4$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+4)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+4)^3}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^4}\right\}$$

$$= \frac{1}{2}t^2e^{-4t} - \frac{4}{6}t^3e^{-4t}$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

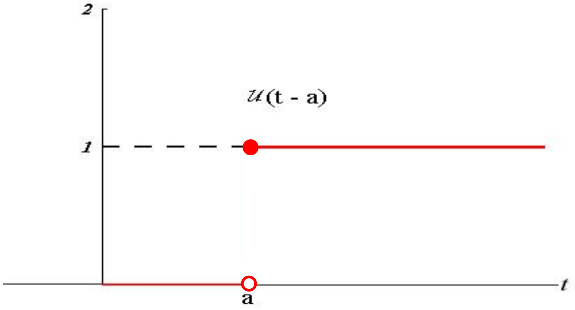

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Unit Step Function Notation

The unit step function is sometimes referred to as the *Heaviside step* function[1]. However, many reserve that name for the version of this function defined on the interval $(-\infty, \infty)$.

- An alternative notations include

$$\mathscr{U}(t-a), \quad u_a(t), \quad u(t-a), \quad \text{and} \quad H(t-a).$$

- Restricting our focus to functions defined on $[0, \infty)$, $f(t) = 1$ and $f(t) = \mathscr{U}(t)$ are indistinguishable.

[1] Named after English mathematician Oliver Heaviside.

Piecewise Defined Functions

Verify that

$a > 0$

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

We want to show that these are equal for all $t \geq 0$. We have to consider the two intervals $0 \le t < a$ and $t \geq a$.

Suppose $0 \le t < a$. Then $\mathscr{U}(t-a) = 0$.

$$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a) = g(t) - g(t)\cdot 0 + h(t)\cdot 0$$

$$= g(t)$$ as expected.

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

Suppose $t \geq a$. Then $\mathscr{U}(t-a) = 1$.

$$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a) = g(t) - g(t)\cdot 1 + h(t)\cdot 1$$

$$= h(t)$$

again, as expected.

Piecewise Defined Functions in Terms of $\mathscr{U}$

Write f on one line in terms of $\mathscr{U}$ as needed

$$f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$$

We can use $\mathscr{U}$ as a switch to turn on and off the pieces of the function along with adding in or subtracting off the pieces.

$$f(t) = \underbrace{e^t}_{\text{on}} - \underbrace{e^t\mathscr{U}(t-2)}_{\text{off}} + \underbrace{t^2\mathscr{U}(t-2)}_{\text{on}} - \underbrace{t^2\mathscr{U}(t-5)}_{\text{off}} + \underbrace{2t\mathscr{U}(t-5)}_{\text{on}}$$

$$f(t) = e^{t}\left(u(t-0) - u(t-2)\right) + t^{2}\left(u(t-2) - u(t-5)\right) + 2t\left(u(t-5) - 0\right)$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

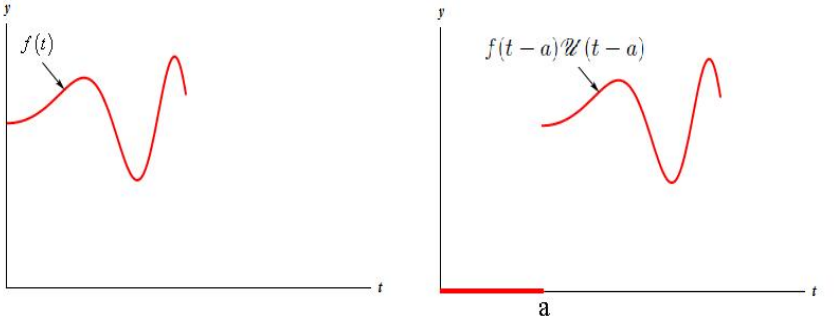

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

In particular,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}.$$

$f(t) = 1$

$\mathscr{L}\{1\} = \frac{1}{s}$

As another example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\{(t-a)^n\mathscr{U}(t-a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$