October 26 Math 2306 sec. 51 Fall 2022

Let's peek at how the Laplace transform will be used to solve ODEs.

If $f(t)$ is defined on $[0, \infty)$, is differentiable, and has Laplace transform $F(s) = \mathscr{L}\{f(t)\}$, then*

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

Use this result to solve the initial value problem

$$y'(t) + 2y(t) = 4, \quad y(0) = 1$$

Let $\mathscr{L}\{y(t)\} = Y(s)$

* See the worksheet 8 from October 24.

$$y'(t) + 2y(t) = 4, \quad y(0) = 1$$

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{4\}$$

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 4\mathcal{L}\{1\}$$

$$sY(s) - y(0) + 2Y(s) = \frac{4}{s}$$

Use $y(0) = 1$ and isolate $Y(s)$

$$sY(s) - 1 + 2Y(s) = \frac{4}{s}$$

$$sY(s) + 2Y(s) = \frac{4}{s} + 1\left(\frac{s}{s}\right)$$

$$(s+2)Y(s) = \frac{4+s}{s}$$

$$Y(s) = \frac{4+s}{s(s+2)}$$

This is the Laplace transform of the solution to the IVP

We need $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Partial fractions

$$\frac{4+s}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \qquad s(s+2)$$

$$4+s = A(s+2) + Bs$$

set $s=0$ $\quad 4 = 2A \Rightarrow A=2$

$s=-2$ $\quad 2 = -2B \Rightarrow B=-1$

$$Y(s) = \frac{2}{s} - \frac{1}{s+2}$$

The solution to the IVP

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\boxed{y(t) = 2 - e^{-2t}}$$

Section 15: Shift Theorems

Theorem: Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We can state this in terms of the inverse transform. If $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

We call this a **translation** (or a **shift**) in s theorem.

Example:

Suppose $f(t)$ is a function whose Laplace transform[1]

$$F(s) = \mathscr{L}\{f(t)\} = \frac{1}{\sqrt{s^2+9}}$$

Evaluate

$$\mathscr{L}\left\{e^{-2t}f(t)\right\} = \frac{1}{\sqrt{(s+2)^2+9}}$$

$$F(s-(-2)) = F(s+2)$$

[1] It's not in our table, but this is an actual function known as a *Bessel function*.

Examples

Evaluate: $\mathscr{L}\left\{te^t\right\} = \dfrac{1}{(s-1)^2}$

$F(s) = \mathscr{L}\{t\} = \dfrac{1}{s^2}$ $\quad a = 1 \qquad F(s-1)$

Evaluate: $\mathscr{L}\left\{t^8e^{-4t}\right\} = \dfrac{8!}{(s+4)^9}$

$F(s) = \mathscr{L}\{t^8\} = \dfrac{8!}{s^9}$, $\quad a = -4 \qquad F(s-(-4)) = F(s+4)$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Partial fractions

$$\frac{-s^2+3s+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Clear fractions

$$-s^2+3s+1 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$-s^2+3s+1 = (A+B)s^2 + (-2A-B+C)s + A$$

$A+B=-1 \qquad B=-1-A=-2$

$-2A-B+C=3 \qquad C=3+B+2A=3-2+2(1)=3$

$A=1$

$$\frac{-s^2+3s+1}{s(s-1)^2}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{(s-1)^2}$$

$$\frac{-s^2+3s+1}{s(s-1)^2}=\frac{1}{s}-\frac{2}{s-1}+\frac{3}{(s-1)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{-s^2+3s+1}{s(s-1)^2}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}=e^{1t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{-s^2+3s+1}{s(s-1)^2}\right\} = 1 - 2e^t + 3e^t t}$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

Heaviside

step

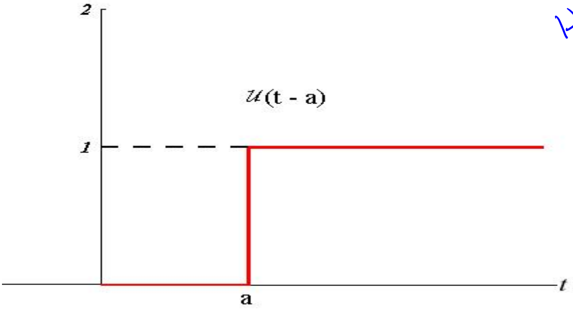

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

Verify that

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$$

$$= g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

We'll consider the two intervals

$0 \le t < a$ and $t \ge a$

Suppose $0 \le t < a$

$$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a) = g(t) - g(t)\cdot 0 + h(t)\cdot 0 = g(t)$$

$$\begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

Suppose $t \geq a$

$$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a) =$$

$g(t) - g(t)(1) + h(t)(1) = g(t) - g(t) + h(t) = h(t)$

So this expansion is $f(t)$ on $0 \leq t < a$

and $t \geq a$.

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

Example $f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$

Rewrite the function f in terms of the unit step function.

$$f(t) = e^t - e^t u(t-2) + t^2 u(t-2) - t^2 u(t-5) + 2t\, u(t-5)$$

Let's verify:

$0 \le t < 2 \qquad f(t) = e^t$

$u(t-2) = 0$

$u(t-5) = 0$

$$f(t) = e^t - e^t u(t-2) + t^2 u(t-2) - t^2 u(t-5) + 2t\, u(t-5)$$

$2 \le t < 5$

$u(t-2) = 1$

$u(t-5) = 0$

$$f(t) = e^t - e^t + t^2 = t^2$$

$t \geq 5$

$u(t-2) = 1$

$u(t-5) = 1$

$$f(t) = e^t - e^t + t^2 - t^2 + 2t = 2t$$