October 26 Math 2306 sec. 52 Fall 2022

Let's peek at how the Laplace transform will be used to solve ODEs.

If $f(t)$ is defined on $[0,\infty)$, is differentiable, and has Laplace transform $F(s)=\mathscr{L}\{f(t)\}$, then*

$$\mathscr{L}\{f'(t)\}=sF(s)-f(0)$$

Use this result to solve the initial value problem

$$y'(t)+2y(t)=4,\quad y(0)=1$$

Let $\mathscr{L}\{y(t)\}=Y(s)$

* See the worksheet 8 from October 24.

$$y'(t) + 2y(t) = 4, \quad y(0) = 1$$

$$\mathcal{L}\{y'(t) + 2y(t)\} = \mathcal{L}\{4\}$$

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 4\mathcal{L}\{1\}$$

$$sY(s) - y(0) + 2Y(s) = \frac{4}{s}$$

$$sY(s) - 1 + 2Y(s) = \frac{4}{s} \qquad \text{using } y(0)=1$$

Isolate $Y(s)$

$$sY(s) + 2Y(s) = \frac{4}{s} + 1$$

$$(s+2)Y(s) = \frac{4+s}{s}$$

$$Y(s) = \frac{4+s}{s(s+2)}$$

this is the Laplace transform of the solution to the IVP

We want $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

we'll do a partial fraction decomp.

$$\frac{4+s}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$

multiply by $s(s+2)$

$$4+s = A(s+2) + Bs$$

Set $s=0$ $\quad 4 = 2A \quad \Rightarrow \quad A = 2$

$s=-2$ $\quad 2 = -2B \quad \Rightarrow \quad B = -1$

$$Y(s) = \frac{2}{s} - \frac{1}{s+2}$$

The solution to the IVP

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\boxed{y(t) = 2 - e^{-2t}}$$

Section 15: Shift Theorems

Theorem: Suppose $\mathscr{L}\left\{f(t)\right\}=F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\}=F(s-a).$$

We can state this in terms of the inverse transform. If $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\}=e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

We call this a **translation** (or a **shift**) in s theorem.

Example:

Suppose $f(t)$ is a function whose Laplace transform[1]

$$F(s) = \mathscr{L}\left\{f(t)\right\} = \frac{1}{\sqrt{s^2+9}}$$

Evaluate

$$\mathscr{L}\left\{e^{-2t}f(t)\right\} = \frac{1}{\sqrt{(s+2)^2+9}}$$

$a = -2 \qquad F(s-(-2)) = F(s+2)$

[1] It's not in our table, but this is an actual function known as a *Bessel function*.

Examples

Evaluate: $\mathscr{L}\left\{te^{t}\right\} = \dfrac{1}{(s-1)^2}$

$\mathcal{L}\{t\} = \dfrac{1!}{s^2} = \dfrac{1}{s^2}$ $\quad a=1, \quad F(s-1)$

Evaluate: $\mathscr{L}\left\{t^{8}e^{-4t}\right\} = \dfrac{8!}{(s+4)^9}$

$F(s) = \mathcal{L}\{t^8\} = \dfrac{8!}{s^9}$, $a=-4$ $\quad F(s-(-4)) = F(s+4)$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Partial fraction

$$\frac{-s^2+3s+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2} \qquad s(s-1)^2$$

Clear fractions

$$-s^2+3s+1 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$-s^2+3s+1 = (A+B)s^2 + (-2A-B+C)s + A$$

$A + B = -1 \quad \Rightarrow \quad B = -1 - A = -2$

$-2A - B + C = 3 \quad \Rightarrow \quad C = 3 + B + 2A = 3 - 2 + 2 = 3$

$A = 1$

$$\frac{-s^2 + 3s + 1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$= \frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}$$

$$\mathscr{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = e^{1t}\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\mathscr{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\} = 1 - 2e^{t} + 3e^{t}t$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

Heaviside

Step

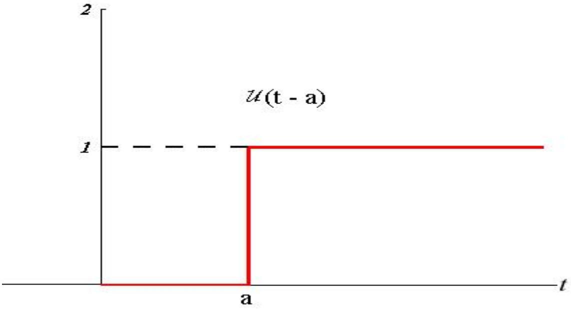

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

$$\mathscr{U}(t-a)=\begin{cases}0, & 0\le t<a\\ 1, & t\ge a\end{cases}$$

Verify that

$$f(t) = \begin{cases} g(t), & 0\le t<a\\ h(t), & t\ge a\end{cases}$$

$$= g(t)-g(t)\mathscr{U}(t-a)+h(t)\mathscr{U}(t-a)$$

We'll consider the two intervals $0\le t<a$ and $a\le t$

Suppose $0\le t<a \implies \mathscr{U}(t-a)=0$

$$g(t)-g(t)\mathscr{U}(t-a)+h(t)\mathscr{U}(t-a) = g(t)-g(t)\cdot 0+h(t)\cdot 0 = g(t)$$

$$\begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

Suppose $t \ge a$

$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a) =$

$g(t) - g(t)(1) + h(t)(1) = h(t)$

Thus $g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$

is equal to $f(t)$ for

all $t \ge 0$.

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

Example $f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$

Rewrite the function f in terms of the unit step function.

$$f(t) = e^t - e^t \mathcal{U}(t-2) + t^2 \mathcal{U}(t-2) - t^2 \mathcal{U}(t-5) + 2t\,\mathcal{U}(t-5)$$

Check:

$0 \le t < 2 \quad \mathcal{U}(t-2) = 0, \quad \mathcal{U}(t-5) = 0$

$f(t) = e^t$

$2 \le t < 5 \quad \mathcal{U}(t-2) = 1$ and $\mathcal{U}(t-5) = 0$

$f(t) = e^t - e^t + t^2 = t^2$

$t \geq 5,\qquad u(t-2) = 1 \qquad u(t-5) = 1$

$$f(t) = e^t - e^t + t^2 - t^2 + 2t = 2t$$