# October 27 Math 2306 sec. 51 Fall 2021

## Section 15: Shift Theorems

**Theorem: (translation in s)** Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number $a$

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

In other words, if $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

# Translation in $t$

We defined the **unit step function**

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

And then we considered the product $f(t-a)\mathscr{U}(t-a)$ for $f$ defined on $[0, \infty)$ and $a > 0$.

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \ge a \end{cases} .$$

## Theorem (translation in $t$)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

Equivalently

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

Find $\mathscr{L}\{\mathscr{U}(t-a)\}$ for some $a > 0$.

$$u(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

By definition

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$= \int_0^a e^{-st} \cdot 0 \, dt + \int_a^\infty e^{-st} \cdot 1 \, dt$$

$$= \int_a^\infty e^{-st}\, dt$$

substitution

set $\tau = t - a$

$d\tau = dt$

$\tau = t - a \Rightarrow t = \tau + a$

when $t = a$, $\tau = a - a = 0$

as $t \to \infty$, $\tau \to \infty$

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-s(\tau+a)}\, d\tau$$

$$e^{-s(\tau+a)}$$

$$= e^{-s\tau} \cdot e^{-sa}$$

$$= e^{-as}\, e^{-s\tau}$$

$$= e^{-as} \int_0^\infty e^{-s\tau}\, d\tau$$

$$= e^{-as}\, \mathcal{L}\{1\}$$

$$= \frac{e^{-as}}{s}$$

# Example

Find the Laplace transform $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}$$

(a) First write $f$ in terms of unit step functions.

$$f(t) = 1 - 1\,\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

$$= 1 + (-1 + t)\,\mathcal{U}(t-1)$$

$$= 1 + (t-1)\,\mathcal{U}(t-1)$$

Note that if $g(t) = t$

then $g(t-1) = t-1$

so $(t-1)u(t-1)$ looks

like $g(t-1)u(t-1)$

where $g(t) = t$.

# Example Continued...

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1 + (t-1)\mathscr{U}(t-1)\}$$

$a = 1$

$$= \mathscr{L}\{1\} + \mathscr{L}\{\underbrace{(t-1)}_{\text{see } t}\mathscr{U}(t-1)\}$$

$$= \frac{1}{s} + e^{-1s}\frac{1}{s^2}$$

$$= \frac{1}{s} + \frac{e^{-s}}{s^2}$$

# A Couple of Useful Results

Another formulation of this translation theorem is

(1) $\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$

$g(t) = g(t+a-a)$

Example: Find $\mathscr{L}\{\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\} = e^{-\frac{\pi}{2}s}\mathscr{L}\{\cos(t+\frac{\pi}{2})\}$

$= e^{-\frac{\pi}{2}s}\mathscr{L}\{-\sin t\} = -e^{-\frac{\pi}{2}s}\frac{1}{s^2+1}$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(t+\frac{\pi}{2}) = \cos t\cos\frac{\pi}{2} - \sin t\sin\frac{\pi}{2} = -\sin t$

# A Couple of Useful Results

The inverse form of this translation theorem is

$$(2) \quad \mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$ $= f(t-2)\mathscr{U}(t-2)$

Here $a=2$, we need to know what $f(t)$ is.

$f(t) = \mathscr{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$ This requires a partial fraction decomp.

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$1 = A(s+1) + Bs$$

set $s=0 \quad 1 = A$ $\Rightarrow$ $A = 1$ and $B = -1$

$s = -1 \quad 1 = -B$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$$

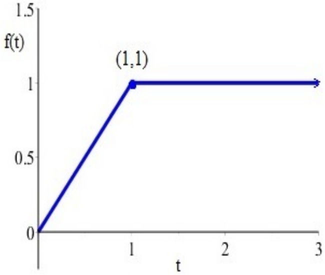
$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 - e^{-t}$$

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = f(t-2)\mathcal{U}(t-2)$$

$$= \left(1-e^{-(t-2)}\right)\mathcal{U}(t-2)$$

# Evaluate the Laplace tranform of the function pictured




Convert f to stacked piecewise notation.

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$$

Figure: Graph of $y = f(t)$. Note: This is exercise 2(b) from section 13 of the workbook.

Write f in terms of unit steps.

$$f(t) = t - t\mathcal{U}(t-1) + 1\mathcal{U}(t-1)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t - t\,u(t-1) + u(t-1)\}$$

$$= \mathcal{L}\{t\} - \mathcal{L}\{t\,u(t-1)\} + \mathcal{L}\{u(t-1)\}$$

$$= \frac{1}{s^2} - e^{-s}\mathcal{L}\{t+1\} + \frac{e^{-s}}{s}$$

$$g(t) = t \Rightarrow g(t+1) = t+1$$

$$= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + \frac{e^{-s}}{s}$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}.$$

# Evaluate

$$\mathscr{L}\left\{(t^2 - e^{3t})^2\right\}$$

$$(t^2 - e^{3t})^2 = t^4 - 2t^2e^{3t} + e^{6t}$$

$$\mathscr{L}\{(t^2 - e^{3t})^2\} = \mathscr{L}\{t^4\} - 2\mathscr{L}\{t^2e^{3t}\} + \mathscr{L}\{e^{6t}\}$$

$$= \frac{4!}{s^5} - 2\frac{2!}{(s-3)^3} + \frac{1}{s-6}$$

$$= \frac{4!}{s^5} - \frac{4}{(s-3)^3} + \frac{1}{s-6}$$

$$\mathscr{L}\{t^2e^{3t}\} = F(s-3) \text{ where } F(s) = \mathscr{L}\{t^2\} = \frac{2!}{s^3}$$