October 27 Math 2306 sec. 51 Spring 2023

Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s,t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_\alpha^\beta G(s,t)\,dt$$

- the result is a function of the remaining variable s, and

- the variable s is treated as a constant while integrating with respect to t.

For Example...

Assume that $s \neq 0$ and $b > 0$. Compute the integral

$$\int_0^b e^{-st}\,dt = \frac{1}{-s} e^{-st} \Big|_0^b$$

$$= \frac{-1}{s} e^{-sb} - \frac{-1}{s} e^{-s(0)}$$

$$= \frac{1}{s} - \frac{1}{s} e^{-sb}$$

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha \int_a^b K(s,t)f(t)\,dt + \beta \int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition:

Let $f(t)$ be piecewise continuous on $[0,\infty)$. The Laplace transform of f, denoted $\mathscr{L}\{f(t)\}$ is given by.

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt. = F(s)$$

We will often use the upper case/lower case convention that $\mathscr{L}\{f(t)\}$ will be represented by $F(s)$. The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Remark 1: The **kernel** for the Laplace transform is $K(s,t) = e^{-st}$.

Remark 2: In general, s is considered a complex variable. We will generally take s to be real, but this will not restrict our use of the Laplace transform.

Limits at Infinity e^{-st}

If $s > 0$, evaluate

$$\lim_{t\to\infty} e^{-st} = 0$$

$-st \to -\infty$

If $s < 0$, evaluate

$$\lim_{t\to\infty} e^{-st} = \infty$$

$-st \to +\infty$

Find[1] the Laplace transform of $f(t) = 1$.

By definition

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt = \int_0^\infty e^{-st} \, dt$$

suppose $s=0$. The integral becomes

$$\int_0^\infty dt = \lim_{b\to\infty} \int_0^b dt = \lim_{b\to\infty} t\Big|_0^b = \lim_{b\to\infty} b = \infty$$

The integral diverges, hence zero is not in the domain of $\mathcal{L}\{1\}$.

For $s \neq 0$,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \, dt = \lim_{b\to\infty} \int_0^b e^{-st} \, dt$$

[1] Unless stated otherwise, the domain for each example is $[0, \infty)$. That is, f is defined for $0 \le t < \infty$.

$$= \lim_{b\to\infty} \frac{1}{-s} e^{-st} \Big|_0^b = \lim_{b\to\infty} \frac{1}{s} - \frac{1}{s} e^{-sb}$$

The limit is finite only when $s > 0$.

For $s > 0$

$$\mathcal{L}\{1\} = \lim_{b\to\infty} \frac{1}{s} - \frac{1}{s} e^{-sb} = \frac{1}{s} - \frac{1}{s}(0) = \frac{1}{s}$$

Hence $\mathcal{L}\{1\} = \frac{1}{s}$ with domain $s > 0$.

Find the Laplace transform of $f(t) = t$.

By definition

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t \, dt$$

If $s=0$, the integral $\int_0^\infty t \, dt$ is divergent.

For $s \neq 0$,

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t \, dt$$

Int. by Parts

$u = t$, $du = dt$

$v = \frac{-1}{s} e^{-st}$ $dv = e^{-st} dt$

$$= \frac{-1}{s} t e^{-st} \Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st} \, dt$$

Convergence requires $s > 0$.

$$= 0 - 0 + \frac{1}{s} \underbrace{\int_0^\infty e^{-st}\, dt}_{\mathcal{L}\{1\}}$$

$$\Rightarrow \mathcal{L}\{t\} = \frac{1}{s} \mathcal{L}\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) \text{ for } s > 0$$

i.e. $\mathcal{L}\{t\} = \frac{1}{s^2}$ for $s > 0$.

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \leq t < 10 \\ 0, & t \geq 10 \end{cases}$$

By def. $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^{10} e^{-st} f(t)\,dt + \int_{10}^\infty e^{-st} f(t)\,dt$

$= \int_0^{10} e^{-st} (2t)\,dt + \int_{10}^\infty e^{-st} (0)\,dt$ 0

For $s=0$, we get

$\int_0^{10} 2t\,dt = t^2 \Big|_0^{10} = 100$

For $s \neq 0$

$$\int_0^{10} e^{-st}(2t)\,dt = \frac{-1}{s}(2t)e^{-st}\Big|_0^{10} + \frac{2}{s}\int_0^{10} e^{-st}\,dt$$

$$= \frac{-2}{s}(10)e^{-s(10)} - 0 + \frac{-2}{s^2}e^{-st}\Big|_0^{10}$$

$$= \frac{-20}{s}e^{-10s} - \frac{2}{s^2}e^{-10s} + \frac{2}{s^2}e^{0}$$

$$\mathcal{L}\{f(t)\} = \begin{cases} 100, & s = 0 \\ \frac{2}{s^2} - \frac{2}{s^2}e^{-10s} - \frac{20}{s}e^{-10s}, & s \neq 0 \end{cases}$$