October 27 Math 2306 sec. 52 Spring 2023

Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s,t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_{\alpha}^{\beta} G(s,t)\,dt$$

- the result is a function of the remaining variable s, and
- the variable s is treated as a constant while integrating with respect to t.

For Example...

Assume that $s \neq 0$ and $b > 0$. Compute the integral

$$\int_0^b e^{-st}\,dt = \frac{1}{-s} e^{-st}\Big|_0^b = \frac{-1}{s} e^{-sb} - \frac{-1}{s} e^{-s(0)}$$

$$= \frac{-1}{s} e^{-sb} + \frac{1}{s}$$

$$= \frac{1}{s} - \frac{1}{s} e^{-sb}$$

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt+\beta\int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition:

Let $f(t)$ be piecewise continuous on $[0,\infty)$. The Laplace transform of f, denoted $\mathscr{L}\{f(t)\}$ is given by.

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt. = F(s)$$

We will often use the upper case/lower case convention that $\mathscr{L}\{f(t)\}$ will be represented by $F(s)$. The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Remark 1: The **kernel** for the Laplace transform is $K(s,t) = e^{-st}$.

Remark 2: In general, s is considered a complex variable. We will generally take s to be real, but this will not restrict our use of the Laplace transform.

Limits at Infinity e^{-st}

If $s > 0$, evaluate

$$\lim_{t\to\infty} e^{-st} = 0$$

$$-st \to -\infty$$

If $s < 0$, evaluate

$$\lim_{t\to\infty} e^{-st} = \infty$$

$$-st \to +\infty$$

Find[1] the Laplace transform of $f(t) = 1$.

By definition $\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt$

If $s=0$, the integral becomes $\int_0^\infty 1 \, dt$

$$\int_0^\infty dt = \lim_{b\to\infty} \int_0^b dt = \lim_{b\to\infty} t \Big|_0^b = \lim_{b\to\infty} b = \infty$$

The integral is divergent, hence zero is not in the domain of $\mathcal{L}\{1\}$.

For $s \neq 0$,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \, dt = \lim_{b\to\infty} \int_0^b e^{-st} \, dt$$

[1] Unless stated otherwise, the domain for each example is $[0, \infty)$. That is, f is defined for $0 \le t < \infty$.

$$= \lim_{b\to\infty} \frac{1}{s} - \frac{1}{s}e^{-sb} = \frac{1}{s} - \frac{1}{s}\cdot 0 = \frac{1}{s}$$

only if $s>0$.

The integral diverges if $s<0$.

$\mathcal{L}\{1\} = \frac{1}{s}$ w/ domain $s>0$.

Find the Laplace transform of $f(t) = t$.

By definition, $\mathcal{L}\{t\} = \int_0^\infty e^{-st}\,t\,dt$

For $s=0$, we get $\int_0^\infty t\,dt$ which diverges. Zero is not in the domain of $\mathcal{L}\{t\}$.

For $s \neq 0$,

$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\,t\,dt$

Int. by parts

$u = t \qquad du = dt$

$v = \frac{-1}{s}e^{-st} \qquad dv = e^{-st}\,dt$

$= \frac{-1}{s}te^{-st}\Big|_0^\infty - \int_0^\infty \frac{-1}{s}e^{-st}\,dt$

$s>0$ is required for convergence

$$= 0 - 0 + \frac{1}{s}\underbrace{\int_0^\infty e^{-st}\,dt}_{\mathcal{L}\{1\}} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

Hence $\mathcal{L}\{t\} = \dfrac{1}{s^2}$ w/ domain $s > 0$.

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

By definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{f(t)\} = \int_0^{10} e^{-st} f(t)\, dt + \int_{10}^\infty e^{-st} f(t)\, dt$$

$$= \int_0^{10} e^{-st} (2t)\, dt + \int_{10}^\infty e^{-st} (0)\, dt$$

For $s=0$, we get $\int_0^{10} 2t\, dt = t^2 \Big|_0^{10} = 100$

For $s \neq 0$ we have

$$\int_0^{10} 2t e^{-st}\,dt = \frac{-2}{s} t e^{-st}\Big|_0^{10} + \int_0^{10} \frac{2}{s} e^{-st}\,dt$$

$$= \frac{-2}{s}(10) e^{-s(10)} - 0 + \frac{2}{s}\left(-\frac{1}{s}\right) e^{-st}\Big|_0^{10}$$

$$= \frac{-20}{s} e^{-10s} - \frac{2}{s^2}\left(e^{-s(10)} - e^{0}\right)$$

$$= \frac{2}{s^2} - \frac{20}{s} e^{-10s} - \frac{2}{s^2} e^{-10s}$$

$$\mathcal{L}\{f(t)\} = \begin{cases} 100, & s = 0 \\ \frac{2}{s^2} - \frac{20}{s} e^{-10s} - \frac{2}{s^2} e^{-10s}, & s \neq 0 \end{cases}$$