October 27 Math 2306 sec. 54 Fall 2021

Section 15: Shift Theorems

Theorem: (translation in s) Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

In other words, if $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

Translation in t

We defined the **unit step function**

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

And then we considered the product $f(t-a)\mathscr{U}(t-a)$ for f defined on $[0,\infty)$ and $a > 0$.

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \ge a \end{cases}.$$

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

Equivalently,

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

Find $\mathscr{L}\{\mathscr{U}(t-a)\}$ $a > 0$

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

By definition

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \int_0^\infty e^{-st}\,\mathcal{U}(t-a)\,dt$$

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$

$$= \int_a^\infty e^{-st}\,dt$$

substitute

set $\tau = t - a$

$d\tau = dt$

$\tau = t - a \Rightarrow t = \tau + a$

when $t = a$, $\tau = a - a = 0$

as $t \to \infty$, $\tau \to \infty$

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \int_0^\infty e^{-s(\tau+a)}\,d\tau$$

$$= e^{-as}\int_0^\infty e^{-s\tau}\,d\tau$$

$$= e^{-as}\,\mathcal{L}\{1\}$$

$$= e^{-as}\left(\frac{1}{s}\right)$$

$$= \frac{e^{-as}}{s}.$$

$$e^{-s(\tau+a)}$$

$$= e^{-s\tau}\cdot e^{-sa}$$

$$= e^{-as}\,e^{-s\tau}.$$

Example

Find the Laplace transform $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$f(t) = 1 - 1\,\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

$$= 1 + (-1 + t)\,\mathcal{U}(t-1)$$

$$= 1 + (t-1)\,\mathcal{U}(t-1)$$

Note that

$(t-1)u(t-1)$ looks like

$h(t-1)u(t-1)$

where $h(t) = t$

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1 + (t-1)\mathcal{U}(t-1)\}$$

$a=1$

$$= \mathscr{L}\{1\} + \mathscr{L}\{(t-1)\mathcal{U}(t-1)\}$$

$$e^{-1s}\mathscr{L}\{t\}$$

$$= \frac{1}{s} + e^{-s}\left(\frac{1}{s^2}\right)$$

$$= \frac{1}{s} + \frac{e^{-s}}{s^2}$$

A Couple of Useful Results

Another formulation of this translation theorem is

(1) $\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$.

$g(t) = g(t+a-a)$

Example: Find $\mathscr{L}\left\{\cos t\, \mathscr{U}\left(t-\frac{\pi}{2}\right)\right\} = e^{-\frac{\pi}{2}s}\mathscr{L}\left\{\cos\left(t+\frac{\pi}{2}\right)\right\}$

$= e^{-\frac{\pi}{2}s}\mathscr{L}\{-\sin t\}$

$= -e^{-\frac{\pi}{2}s}\mathscr{L}\{\sin t\} = -e^{-\frac{\pi}{2}s}\frac{1}{s^2+1^2}$

$= \dfrac{-e^{-\frac{\pi}{2}s}}{s^2+1}$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$\cos\left(t+\frac{\pi}{2}\right) = \cos t \cos\frac{\pi}{2} - \sin t \sin\frac{\pi}{2} = -\sin t$$

0 $\quad$ 1

A Couple of Useful Results

The inverse form of this translation theorem is

$$(2) \quad \mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\} = f(t-2)\mathcal{U}(t-2)$

we need to find $f(t)$.

$f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$ This requires partial fractions

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$1 = A(s+1) + Bs$

Set $s=0 \qquad 1 = A \quad \Rightarrow \quad A = 1$

$s=-1 \qquad 1 = -B \quad \Rightarrow \quad B = -1$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

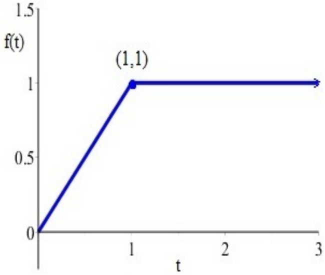
$$= 1 - e^{-t}$$

$$f(t) = 1 - e^{-t}$$

Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\} = f(t-2)\mathcal{U}(t-2)$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = \left(1 - e^{-(t-2)}\right)\mathcal{U}(t-2)$$

Evaluate the Laplace tranform of the function pictured



express f algebraically

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$$

Figure: Graph of $y = f(t)$. Note: This is exercise 2(b) from section 13 of the workbook.

$$f(t) = t - t\,\mathcal{U}(t-1) + 1\,\mathcal{U}(t-1)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t - t\mathcal{U}(t-1) + \mathcal{U}(t-1)\}$$

$$= \mathcal{L}\{t\} - \mathcal{L}\{t\,\mathcal{U}(t-1)\} + \mathcal{L}\{\mathcal{U}(t-1)\}$$

$$= \frac{1}{s^2} - e^{-1s}\mathcal{L}\{t+1\} + \frac{e^{-1s}}{s}$$

$$= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + \frac{e^{-s}}{s}$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

$g(t) = t \quad a = 1 \quad g(t+1) = t+1$