# October 27 Math 3260 sec. 51 Fall 2025

## 4.3.3 Basis as a Subset of a Spanning Set

### Dimension

Let $S$ be a subspace of $R^n$. The **dimension** of $S$ will be denoted $\dim(S)$.

- If $S = \{\vec{0}_n\}$, then $S$ does not have a basis and $\dim(S) = 0$.
- If $S$ contains any nonzero vectors, then $\dim(S) = k$, where $k$ is the number of vectors in any basis of $S$.

# 4.3.3 Basis as a Subset of a Spanning Set

**Lemma**

Let $T = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of $k$ vectors in $R^n$, with $k \geq 2$, and let $S = \mathrm{Span}(T)$. If one of the vectors, say $\vec{v}_i$ in $T$, is a linear combination of the other vectors in $T$, then the set obtained from $T$ by removing $\vec{v}_i$ spans $S$.

This is saying that if a set is linearly dependent, it is possible to remove vector(s) without changing the subspace spanned by that set.

**Examples:**

$$\mathrm{Span}\{\langle 1,0\rangle, \langle -3,0\rangle\} = \mathrm{Span}\{\langle 1,0\rangle\}$$

$$\mathrm{Span}\{\langle 5,0,4\rangle, \langle 1,0,0\rangle, \langle 0,0,2\rangle\} = \mathrm{Span}\{\langle 1,0,0\rangle, \langle 0,0,2\rangle\}$$

# 4.3.3 Basis as a Subset of a Spanning Set

**Lemma**

Let $T = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors in $R^n$ and $S = \mathrm{Span}(T)$. If $T$ contains at least one nonzero vector, then there exists a subset of $T$ that is a basis for $S$.

This tells us that if we have a spanning set, we can whittle it down to a basis if we throw out vectors that are linear combinations of the others.

Unfortunately, the theorem doesn't tell us how to figure out which vectors to remove—it just tells us it's possible to get a basis by culling.

**Lemma**

**The set of pivot column vectors of a matrix $A$ is linearly independent.**

# 4.4 Bases for the Column & Row Spaces of a Matrix

**Theorem**

Let $A$ be an $m \times n$ matrix that is not the zero matrix. Then the pivot columns of $A$ form a basis for $\mathcal{CS}(A)$.

**Remark:** This says that the pivot columns of $\boxed{A}$ form a basis for $\mathcal{CS}(A)$.

We can use row reduction to find pivot columns, but the vectors in our basis come from the original matrix!

**Caution:** $\mathcal{CS}(A)$ is generally NOT the same as $\mathcal{CS}(\text{rref}(A))$.

# Example

$$A = \begin{bmatrix} 1 & 3 & 2 & -2 & -12 & 1 & 41 \\ 2 & 6 & 1 & 0 & 1 & 0 & 12 \\ -1 & -3 & 1 & 3 & 7 & 1 & -1 \\ 0 & 0 & 2 & -1 & -10 & 0 & 30 \\ 3 & 9 & 2 & 1 & 4 & 1 & 23 \end{bmatrix}, \quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 & 12 \\ 0 & 0 & 0 & 1 & 4 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Identify a basis for $\mathcal{CS}(A)$.

From the rref, the pivot columns are 1, 3, 4, and 6.

A basis for CS(A) is

$\{ \langle 1, 2, -1, 0, 3 \rangle, \langle 2, 1, 1, 2, 2 \rangle, \langle -2, 0, 3, -1, 1 \rangle, \langle 1, 0, 1, 0, 1 \rangle \}.$

# Example

$$A = \begin{bmatrix} 1 & 3 & 2 & -2 & -12 & 1 & 41 \\ 2 & 6 & 1 & 0 & 1 & 0 & 12 \\ -1 & -3 & 1 & 3 & 7 & 1 & -1 \\ 0 & 0 & 2 & -1 & -10 & 0 & 30 \\ 3 & 9 & 2 & 1 & 4 & 1 & 23 \end{bmatrix}, \quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 & 12 \\ 0 & 0 & 0 & 1 & 4 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Verify that $\mathrm{Col}_5(A) = 2\,\mathrm{Col}_1(A) - 3\,\mathrm{Col}_3(A) + 4\,\mathrm{Col}_4(A)$.

$$2\langle 1,2,-1,0,3\rangle + (-3)\langle 2,1,1,2,2\rangle + 4\langle -2,0,3,-1,1\rangle$$

$$= \langle 2,4,-2,0,6\rangle + \langle -6,-3,-3,-6,-6\rangle + \langle -8,0,12,-4,4\rangle$$

$$= \langle -12,1,7,-10,4\rangle$$

A linear dependence relation between columns of rref(A) is obvious, but it turns out that the columns of A have the same linear dependence relation.

# Row Space

**Theorem**

If $A$ and $B$ are row equivalent matrices, then $\mathcal{RS}(A) = \mathcal{RS}(B)$.

This means that the row space of a matrix $A$ is the same subspace as the row space of any echelon form that is row equivalent to $A$. In other words, row operations don't change the row space.

**Corollary**

Let $A$ be an $m \times n$ matrix that is not the zero matrix. Then the nonzero rows of $\text{rref}(A)$ form a basis for $\mathcal{RS}(A)$.

**Remark:** This says we can use the actual rows of $\text{rref}(A)$—not $A$ itself—as basis elements for $\mathcal{RS}(A)$

# Bases for Fundamental Subspaces

Given $m \times n$ matrix $A$ that is not the zero matrix:

- Set up $\left[A \mid \vec{0}_m\right]$ and row reduce to $\left[\text{rref}(A) \mid \vec{0}_m\right]$.
- Identify the pivot columns from $\text{rref}(A)$ and use those pivot columns to form a basis for $\mathcal{CS}(A)$.
- Take the nonzero rows of $\text{rref}(A)$ to form a basis for $\mathcal{RS}(A)$.
- Use $\text{rref}(A)$ to deduce the relationship between basic and free variables, and use the factoring process to obtain a basis for $\mathcal{N}(A)$. If $\mathcal{N}(A) = \{\vec{0}_n\}$, then $\mathcal{N}(A)$ doesn't have a basis.
- If a basis for $\mathcal{N}(A^T)$ is desired, use $\left[\text{rref}(A^T) \mid \vec{0}_n\right]$ and the factoring process to obtain a basis.

# Find Bases & Dimensions of $\mathcal{RS}(A)$, $\mathcal{CS}(A)$ and $\mathcal{N}(A)$

$$A = \begin{bmatrix} 2 & 6 & 0 & -2 & -4 \\ 1 & 3 & 1 & -4 & -17 \\ -1 & -3 & -1 & 4 & 17 \\ 2 & 6 & 1 & 0 & 1 \end{bmatrix} \qquad \left[\text{rref}(A) \mid \vec{0}_4\right] = \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

A basis for $RS(A)$ comes from nonzero rows of rref $(A)$. A basis for $RS(A)$ is

$$\{ \langle 1, 3, 0, 0, 2 \rangle, \langle 0, 0, 1, 0, -3 \rangle, \langle 0, 0, 0, 1, 4 \rangle \}.$$

$$\dim(RS(A)) = 3.$$

$$[\mathrm{rref}(A) \mid \vec{0}_4] = \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad A = \begin{bmatrix} 2 & 6 & 0 & -2 & -4 \\ 1 & 3 & 1 & -4 & -17 \\ -1 & -3 & -1 & 4 & 17 \\ 2 & 6 & 1 & 0 & 1 \end{bmatrix}$$

For $CS(A)$, we take the pivot column vectors from $A$. A basis for $CS(A)$ is

$$\{\langle 2,1,-1,2\rangle, \langle 0,1,-1,1\rangle, \langle -2,-4,4,0\rangle\}.$$

$$\dim(CS(A)) = 3$$

For $N(A)$, consider $A\vec{x} = \vec{0}_4$.

For $\vec{x} = \langle x_1, x_2, x_3, x_4, x_5\rangle$, $x_1, x_3, x_4$ are basis

$x_2$ and $x_5$ are free

$$[\text{rref}(A) \mid \vec{0}_4] = \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -3x_2 - 2x_5$

$x_3 = 3x_5$

$x_4 = -4x_5$

$x_2, x_5$ free

$\vec{x} = \langle -3x_2 - 2x_5, x_2, 3x_5, -4x_5, x_5 \rangle$

$= x_2 \langle -3, 1, 0, 0, 0 \rangle + x_5 \langle -2, 0, 3, -4, 1 \rangle$

A basis for $\mathcal{N}(A)$ is

$\{ \langle -3, 1, 0, 0, 0 \rangle, \langle -2, 0, 3, -4, 1 \rangle \}$

$\dim(\mathcal{N}(A)) = 2.$

$$A = \begin{bmatrix} 2 & 6 & 0 & -2 & -4 \\ 1 & 3 & 1 & -4 & -17 \\ -1 & -3 & -1 & 4 & 17 \\ 2 & 6 & 1 & 0 & 1 \end{bmatrix}, \quad \text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three row vectors of $\text{rref}(A)$ form a basis for $\mathcal{RS}(A)$. Explain why the first three row vectors of $A$ **do not** form a basis for $\mathcal{RS}(A)$.

The 1st 3 rows don't form a linearly independent set.

Note that the third row vector is -1 times the second row vector. The first several row vectors could be linearly dependent without it being so obvious as it is in this example.

## Row Operations & Linear Dependence Relations

- Elementary row operations preserve linear dependence relations between columns but change the column space.

- Elementary row operations preserve the row space but change linear dependence relations between the rows.

### Important Observations

- The basis elements for the column space **come from** $A$ not from $\text{rref}(A)$.
- The basis elements for the row space **come from** $\text{rref}(A)$ not from $A$.
- The method we've been using all along to characterize solutions to $A\vec{x} = \vec{0}_m$ gives us a basis for the null space.

# 4.5 The Fundamental Theorem of Linear Algebra

**Dimensions of $\mathcal{CS}(A)$, $\mathcal{RS}(A)$ & $\mathcal{N}(A)$**

For $m \times n$ matrix $A$,

$$\dim(\mathcal{CS}(A)) = \text{ the number of pivot columns of } A.$$

$$\dim(\mathcal{N}(A)) = \text{ the number of non-pivot columns of } A.$$

$$\dim(\mathcal{RS}(A)) = \text{ the number of pivot columns of } A.$$

**Remark:** A basis for the row space is coming from the rows, but each pivot position is in a row and a column. So the number of nonzero rows of $\mathrm{rref}(A)$ has to match the number of pivot columns.

# Rank & Nullity

**Definition: Rank**

The **rank** of a matrix $A$, denoted $\mathrm{rank}(A)$, is the dimension of the column space of $A$.

We also have a special name for the dimension of the null space of a matrix. We call this the nullity.

**Definition: Nullity**

The **nullity** of a matrix $A$, denoted $\mathrm{nullity}(A)$, is the dimension of the null space of $A$.

## The Fundamental Theorem of Linear Algebra

Let $A$ be an $m \times n$ matrix. Then

1. $\text{rank}(A) = \dim\big(\mathcal{CS}(A)\big) = \dim\big(\mathcal{RS}(A)\big)$.
2. $\text{rank}(A) + \text{nullity}(A) = n$.
3. Every vector $\vec{x}$ in $\mathcal{RS}(A)$ is orthogonal to every vector $\vec{y}$ in $\mathcal{N}(A)$, and similarly, every vector $\vec{u}$ in $\mathcal{CS}(A)$ is orthogonal to every vector $\vec{v}$ in $\mathcal{N}(A^T)$.

Part 2. of the FTLA is often called the *rank-nullity theorem*. It follows from the observation that

$$\begin{array}{rl} & \text{the number of pivot columns of } A \\ + & \text{the number of non-pivot columns of } A \\ \hline = & \text{the total number of columns of } A. \end{array}$$

# Example

Suppose $A$ is a $12 \times 20$ matrix.

1. If $\text{rank}(A) = 9$, how many free variables are there for $A\vec{x} = \vec{0}_{12}$?

rank(A) + nullity(A) = n    here    n = 20

9 + nullity(A) = 20 ⇒ nullity(A) = 11

A has 11 non pivot columns, so $A\vec{x} = \vec{0}_{12}$

has 11 free variables.

# Example

Suppose $A$ is a $12 \times 20$ matrix.

2. If $\text{rref}(A)$ has seven nonzero rows, what is $\text{nullity}(A^T)$?

$A^T$ is $20 \times 12$

$\text{rank}(A^T) + \text{nullity}(A^T) = 12$

$\text{rref}(A)$ has 7 nonzero rows $\Rightarrow \dim(RS(A)) = 7$

$\dim(CS(A^T)) = \dim(RS(A)) = 7 = \text{rank}(A^T)$

$\Rightarrow \text{nullity}(A^T) = 12 - 7 = 5$

# Example

Let $B = \begin{bmatrix} 1 & 3 & 1 & 0 & 2 \\ 2 & 2 & -2 & 4 & 0 \\ 3 & 1 & -5 & 8 & 1 \end{bmatrix}$. Find the rank and nullity of $B$.

$$\text{rref}(B) = \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

B has 3 pivot columns and 2 nonpivot columns

$\text{rank}(B) = $ 3 and $\text{nullity}(B) = $ 2.

# Example

Suppose $A$ is an $n \times n$ invertible matrix. What is $\mathrm{rank}(A)$ and what is $\mathrm{nullity}(A)$?

If A is invertible, it's set of column vectors are linearly independent. So all n columns are pivot columns

$\mathrm{rank}(A) = n$

$\mathrm{nullity}(A) = 0$

The result on the nullity makes sense. $A\vec{x}=\vec{0}_n$ will have only the trivial solution, so $\mathcal{N}(A)=\{\vec{0}_n\}$. It has dimension zero.